GEOMETRY OF SUBMANIFOLDS

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*This is to certify that the dissertation entitled **"Geometry of Submanifolds"** has been carried out by **Ms. Nargis Jamal** under my supervision and the work is suitable for submission for the award of the degree of Master of Philosophy in Mathematics.*

(Khalid Ali Khan)
Supervisor

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Nargis Jamal

Preface

The study of geometry of submanifolds of an almost Hermition manifold is one of the most fascinating topics in differential geometry. The submanifolds of an almost Hermition manifold present an interesting geometry as the almost complex structure J on the ambient manifold transforms a vector to a vector perpendicular to it, the natural outcome of which are three typical classes of submanifolds, namely holomorphic or invariant submanifolds (also known as almost complex submanifolds) totally real or anti invariant submanifolds and slant submanifolds (cf., [9], [13] [42] etc.).

The holomorphic and totally real submanifolds M are characterized by the conditions $J(T_xM) \subseteq T_xM$ and $J(T_xM) \subseteq T_xM^{\perp}$ respectively, where T_xM denotes the tangent space and $T_xM^{\perp}$, the normal space at $x \in M$. Slant submanifolds were introduced by B.Y Chen in 1990 and are defined by the constant angle θ (known as Wirtinger angle) between $J(T_xM)$ and T_xM [9]. Obviously if θ is 0 then submanifold is holomorphic and if θ is $\pi/2$ then it is totally real. The notion of CR-submanifolds was introduced by A. Bejancu in 1978 as a generalization to the holomorphic and totally real submanifolds. A real submanifold M of an almost Hermition manifold is called CR-submanifold if there exists a differentiable dtstribution D on M satisfying (i) $J(D_x) = D_x$ and (ii) $J(D_x^{\perp}) \subseteq T_xM^{\perp}$, for each $x \in M$, where $D^{\perp}$ is the complementary orthogonal distribution to D.

CR-submanifolds are an active area of research for the past thirty years and play important role in many diverse areas of differential geometry relativity as well as in mechanics [3], [10]. Integrability of the distributions gives rise to the notion of CR-product submanifolds, which are those CR-submanifolds that are locally Riemannian product of the leaves of D and $D^{\perp}$. A lot of research has been done on CR-product submanifolds and characterizations are found for a CR-submanifold to become a CR-product submanifold (cf., [11], [12], [26] etc.). Moereover it is proved that there do not exist non-trivial CR-products in complex hyperbolic spaces [37]. It was also found that S^6 does not admit non-trivial CR-product submanifolds [37]. CR-products however are obtained in complex projective spaces naturally via Segre imbedding.

Bishop O'Neill [8] in 1969 introduced warped product manifolds as a generalization to Riemannian product manifolds. Easiest examples of warped product manifolds are surfaces of revolution. Formally, a warped product manifold is defined as: Let B and F be two Riemannian manifolds with metrics g_B and g_F respectively and f is a positive differentiable function on B. The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with Riemannian metric $g = g_B + f^2 g_F$. The function f is called warping function of the warped product manifold M.The study of warped products got impetus when B.Y Chen studied warped product CR-submanifolds of a Kaehler manifold [16], [17].
The CR-product and warped product submanifolds form the main theme of this dissertation.

The dissertation comprises of five chapters and each chapter is divided into various sections. The mathematical relations obtained in the text have been labeled with double decimal numbering. The first figure denotes the chapter number, second represents the section and the third points out the number of the defination, equation, proposition, corollary or the theorem, as the case may be for example, theorem 3.2.1, refers to first theorem of second article in the third chapter.

The first chapter is introductry and contains those definitions and results which are relevant for the subsequent ones. Moreover, this serves the purpose of making the dissertation as self contained as possible and fixes up the terminology for the forthcoming chapters.

Chapter II deals with the CR-product submanifolds of Kaehler manifolds. We discuss first integrability conditions of the distributuions D and $D^{\perp}$ on a CR-submanifold of an almost Hermition manifold in terms of Nijenhuis tensor and the conditions for these to become CR-product submanifolds proved by Sato, Urbano and others. We then provide some results about CR and CR-product submanifolds proved by B.Y Chen, Chen and Blair in [11], [12], [7] when the ambient manifold is a Kaehler manifold.

In chapter III we extend the results of second chapter by taking the ambient manifold to the more general setting of nearly Kaehler and almost Kaehler manifold. This chapter includes the results by Khan, K.A., Khan,V.A and Hussain, S.I. who introduced tensors $\mathcal{P}$ and Q on the tangent bundle of the submanifold in [26] and used them to prove and extend Chen's characterization of CR-product submanifolds.

In chapter IV, we discuss the warped product CR-submanifolds of Kaehler manifolds. The warped product CR-submanifolds can be defined two ways $N_T \times_f N_{\perp}$ and $N_{\perp} \times_f N_T$. In the first case it was found [16] that they are no different then the CR-product submanifolds. We also in this chapter describe some examples of warped product manifolds.

In the last chapter, chapter V, we have studied some of the recent progress on warped product submanifolds. We first discussed non-existence of the warped product semi-slant submanifolds of Kaehler manifolds proved by B. Sahin [36] and then gave the generalizations of some of the results such as the characterization of the warped product CR-submanifolds in Kaehler manifolds by B.Y. Chen to the nearly Kaehler settings by V.A. Khan , K.A. Khan and others.

In the end we have given a bibliography which by no means is exhaustive on the subject, but contains only those references which are referred in the text.

CHAPTER I

INTRODUCTION

1.1 Introduction.

The main aim of this chapter is to introduce basic concepts, preliminary notions and some fundamental results that are required for the development of the subject in the present dissertation. Thus in this chapter we have given a brief resume of some of the results in the geometry of almost Hermitian manifolds and the allied structures and the geometry of submanifolds of these manifolds. Although most of these results are readily available in review articles and some in standard books e.g., Nomizu and Kobayashi [29] B.Y.Chen [10] , Yano [42], and D.E.Blair [7] nevertheless, we have collected them here to set up our terminology and for ready references.

1.2 Structures on C^∞-Manifolds.

We can discover the geometry of a differentiable manifold by knowing a Riemannian metric on it. Further refined information can be had through additional structures on the manifold, for example almost complex, almost Kaehler, nearly Kaehler and almost contact structures etc.,([23], [25]). In this section we briefly discuss some of these structures.

By a *Riemannian metric* g on a manifold M, we mean a map $p \to g_p$ where g_p is a positive definite inner product on $T_p(M)$. We require this map to be smooth in the sense that the function

$$p \to g_{ij}(p) = g_p(\frac{\partial}{\partial x_i}|_p, \frac{\partial}{\partial x_j}|_p)$$

is smooth for all i, j for any chart (U, x) on M. This smoothness condition is same as requiring that for all vector fields X, Y on M, $p \to g_p(X_p, Y_p)$ is smooth. On a paracompact manifold, there exist a smooth Riemannian metric.

On a Riemannian manifold (M, g) with a connection ∇ the Koszual formula is defined as

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z)$$

$$+ g([Z, X]) + g(X, [Z, Y]) \tag{1.2.1}$$

In what follows, we shall take a differentiable manifold that is connected and paracompact, so that it can always be endowed with a Riemannian metric g and a Riemannian connection ∇.

An *almost complex structure* on a real differentiable manifold $\bar{M}$ is a tensor field J which is at every point $p \in \bar{M}$, an endomorphism of the tangent space $T_p(M)$ such that $J^2 = -I$ where I denotes the identity transformation. A manifold with a fixed almost complex structure is called an *almost complex manifold*. On an almost complex manifold, there always exist a Riemannian metric g consistent with the almost complex structure J i.e., satisfying

$$g(JU, JV) = g(U, V) \tag{1.2.2}$$

for all $U, V \in T(\bar{M})$. Here $T(\bar{M})$ denotes the tangent bundle of the manifold $\bar{M}$. By virtue of which g is called a *Hermitian metric*. An almost complex manifold (resp. a complex manifold) equipped with a Hermitian metric is called an *almost Hermitian manifold* (resp. a *Hermitian manifold*).

Analogous to the almost complex structure J, there is defined a 2-form which plays an important role in the geometry as well as in the mechanics on the manifold [31]. We describe it as follows:

Definition (1.2.1). A *symplectic form* on a real vector space V of dimension n is a non-degenerate exterior 2-form Φ of rank n. If V admits a symplectic form Φ then we say that Φ defines a symplectic structure on V or that $(V,\ \Phi)$ is a *symplectic vector space*.

A symplectic structure on a manifold $\bar{M}$ is defined by a choice of a differentiable 2-form Φ satisfying the following two conditions

(i) For all $p \in M$, Φ_p is non-degenerate.

(ii) Φ is closed, that is $d\Phi = 0$.

One can define a Kaehler manifold using the fundamental 2-form Φ, almost complex structure J and the Riemannian metric g as follows:

Let Φ denote the fundamental 2-form associated with the Hermitian metric g on $\bar{M}$ i.e.,

$$\Phi(U,V) = g(U,JV) \tag{1.2.3}$$

for all vector fields U, V on M. Since g is invariant under J so is Φ, i.e.,

$$\Phi(JU,JV) = \Phi(U,V). \tag{1.2.4}$$

The almost complex structure J is not in general parallel with respect to the Riemannian connection $\bar{\nabla}$ on $\bar{M}$, defined by the Hermitian metric g. In fact, we have the following formula

$$\begin{aligned} 4g((\bar{\nabla}_U J)V,W) = {}& 6d\Phi(U,JV,JW) - 6d\Phi(U,V,W) \\ & + g(S(V,W),JU) \end{aligned} \tag{1.2.5}$$

where S is the Nijenhuis tensor of J defined by

$$S(U,V) = 2\{[JU,JV] - [U,V] - J[U,JV] - J[JU,V]\}. \tag{1.2.6}$$

It is easy to verify that S satisfies

$$S(JU,V) = S(U,JV) = -JS(U,V). \tag{1.2.7}$$

It is well known that vanishing of the tensor $S(U,V)$ is the necessary and sufficient condition for an almost complex manifold to be a complex manifold [25].

If we extend the Riemannian connection $\bar{\nabla}$ to be a derivative on the tensor algebra of $\bar{M}$, then we have the following formulae

$$(\bar{\nabla}_U J)V = \bar{\nabla}_U JV - J\bar{\nabla}_U V, \tag{1.2.8}$$

$$(\bar{\nabla}_U \Phi)(V,W) = g((\bar{\nabla}_U J)V, W). \tag{1.2.9}$$

Definition (1.2.2). A Hermitian metric on an almost complex manifold is called a *Kaehler metric* if the fundamental 2-form Φ is closed. A complex manifold equipped with a Kaehler metric is said to be a *Kaehler manifold*. In other words, an almost complex manifold $\bar{M}$ is Kaehler if

$$(\bar{\nabla}_U J)V = 0 \tag{1.2.10}$$

or equivalently,

$$\bar{\nabla}_U JV = J\bar{\nabla}_U V$$

for all U, V in $T(\bar{M})$. In this case the connection $\bar{\nabla}$ on $\bar{M}$ is said to be a *Kaehler connection*.

A Hermitian manifold $\bar{M}$ is said to be *nearly Kaehler* if

$$(\bar{\nabla}_U J)V + (\bar{\nabla}_V J)U = 0 \tag{1.2.11}$$

for all U, V in $T(\bar{M})$ and is called *almost Kaehler* if $(\bar{\nabla}_U J)U + (\bar{\nabla}_{JU} J)JU = 0$ for all U in $T(\bar{M})$. Following figure depicts the relationships between these classes of manifolds.

$$\begin{array}{ccc} K & \subseteq & AK \\ \cap & & \\ NK & & \end{array} \qquad \& \qquad K \subseteq H, \quad K = AK \cap NK$$

Figure(1.2.1)

Definition (1.2.3). A Kaehler manifold $\bar{M}$ is called a *complex-space-form* if it has constant holomorphic sectional curvature. We denote by $\bar{M}^m(c)$ (or simply $\bar{M}(c)$) a complex m-dimensional complex-space-form of constant holomorphic sectional curvature c. The curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(U,V)W = \frac{c}{4}\{g(V,W)U - g(U,W)V + g(JV,W)JU) \\ -g(JU,W)JV + 2g(U,JV)JW\} \end{aligned} \tag{1.2.12}$$

for any vector fields U, V, W on $\bar{M}$.

Let $\bar{M}$ be a $(2m+1)$-dimensional differentiable manifold. An *almost contact structure* $(\phi,\ \xi,\ \eta)$ on $\bar{M}$ consist of a tensor field ϕ of type $(1,\ 1)$, a vector field ξ and a 1-form η which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1 \tag{1.2.13}$$

where I denotes the identity tensor on $\bar{M}$. A manifold $\bar{M}$ with an almost contact structure is called an *almost contact manifold.* The condition (1.2.13) also imply

$$\phi(\xi) = 0 \qquad \text{and} \qquad \eta \circ \phi = 0. \tag{1.2.14}$$

Now, suppose on $\bar{M}$ there is given a Riemannian metric tensor field g which satisfies

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V) \tag{1.2.15}$$

for any vector fields U, V on $\bar{M}$. Then the structure $(\phi,\ \xi,\ \eta, g)$ is said to be an *almost contact metric structure* on $\bar{M}$. In this case, it is easy to check that

$$g(U, \xi) = \eta(U) \tag{1.2.16}$$

for any vector field U on $\bar{M}$.

An almost contact metric structure is called a *contact metric structure* if

$$d\eta = \Phi$$

where Φ is the fundamental 2-form defined by

$$\Phi(U, V) = g(U, \phi V).$$

In this case for any vector field U on $\bar{M}$, we have

$$\bar{\nabla}_U \xi = -\phi U - \phi h U \tag{1.2.17}$$

where $h = \frac{1}{2} L_\xi \phi$, $L_\xi \phi$ being the Lie derivative of ϕ with respect to ξ. The operator h satisfies

$$g(hU, V) = g(U, hV), \qquad \phi \circ h = -h \circ \phi. \tag{1.2.18}$$

If the characteristic vector field ξ of the contact metric structure $(\phi,\ \xi,\ \eta, g)$ is killing with respect to g, we refer to this structure as a *K-contact structure.* In this case, (1.2.17) becomes

$$\phi U = -\bar{\nabla}_U \xi. \tag{1.2.19}$$

By a *Sasakian manifold,* we mean a contact metric manifold which is normal i.e.,

$$S_\phi + 2d\eta \otimes \xi = 0 \tag{1.2.20}$$

where S_ϕ is the Nijenhuis tensor of the tensor field ϕ. Every Sasakian manifold is a K-contact manifold and the following equation holds in a *Sasakian manifold,*

$$(\bar{\nabla}_U\phi)V = g(U,V)\xi - \eta(V)U. \tag{1.2.21}$$

1.3 Submanifold Theory.

If an n-dimensional differentiable manifold M admits an immersion

$$f : M \hookrightarrow \bar{M}$$

into an m-dimensional differentiable manifold $\bar{M}$, then M is said to be a *submanifold* of $\bar{M}$. Naturally $n \leq m$. If $\bar{M}$ is a Riemannian manifold with a Riemannian metric g, then M also admits a Riemannian metric induced from $\bar{M}$ which is denoted by the same symbol g. The immersion f is said to be an *isometric immersion* if the differentiable map $f_* : TM \hookrightarrow T\bar{M}$ preserve the Riemannian metric, that is for $U, V \in T(M)$

$$g(f_*U, f_*V) = g(U,V). \tag{1.3.1}$$

When only local questions are involved, we shall identify $T(M)$ with $f_*T(M)$ through the isomorphism f_*. Hence, a tangent vector in $T(\bar{M})$ tangent to M, shall mean tangent vector which is the image of an element in $T(M)$ under f_*. More generally, a C^∞-cross section of the restriction of $T(\bar{M})$ on M shall be called a vector field of $\bar{M}$ on M. Those tangent vectors of $T(\bar{M})$, which are normal to $T(M)$ form the normal bundle $T^\perp(M)$ of M. Hence for every point $p \in M$, the tangent space $T_{f(p)}(\bar{M})$ of $\bar{M}$ admits the following decomposition

$$T_{f(p)}(\bar{M}) = T_p(M) \oplus T_p^\perp(M).$$

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^{\perp}$ on TM and on the normal bundle $T^{\perp}(M)$ respectively governed by the Gauss and Weingarten formulae v.i.z.

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{1.3.2}$$

$$\bar{\nabla}_U N = A_N U + \nabla^{\perp}_U N \tag{1.3.3}$$

where U, V are tangent vector fields on M and $N \in T^{\perp}(M)$. h and A_N are second fundamental forms and are related by

$$g(h(U, V), N) = g(A_N U, V). \tag{1.3.4}$$

Looking into the Gauss formula, we observe that, one can classify the submanifolds, putting conditions on h as follows

Definition (1.3.1) [10]. A submanifold for which the second fundamental form h is identically zero is called a *totally geodesic submanifold.*

Definition (1.3.2)[10]. A submanifold is called *totally umbilical* if its second fundamental form h satisfies

$$h(U, V) = g(U, V)H \tag{1.3.5}$$

where $H = \frac{1}{n}(\text{trace of } h)$, is called the *mean curvature vector.*

Definition (1.3.3) [10]. A submanifold is called *minimal* if the mean curvature vector vanishes identically. i.e, $H = 0$.

For the second fundamental form h, we define the covariant differentiation $\bar{\nabla}$ with respect to the connection in $TM \oplus T^{\perp}M$ by

$$(\bar{\nabla}_U h)(V, W) = \nabla^{\perp}_U h(V, W) - h(\nabla_U V, W) - h(V, \nabla_U W) \tag{1.3.6}$$

for any vector field U, V and W tangent to M.

The equations of Gauss, Coddazi and Ricci are then given by

$$R(U,V;W,Z) = \bar{R}(U,V,W,Z) + g(h(U,Z),h(V,W)) \\ - g(h(U,W),h(V,Z)), \qquad (1.3.7)$$

$$(\bar{R}(U,V)W)^{\perp} = (\bar{\nabla}_U h)(V,W) - (\bar{\nabla}_V h)(U,W), \qquad (1.3.8)$$

$$\bar{R}(U,V,N_1,N_2) = R^{\perp}(U,V,N_1,N_2) - g([A_{N_1},A_{N_2}]U,V), \quad (1.3.9)$$

where $\qquad R(U,V;W,Z) = g(R(U,V)W,Z).$

In (1.3.8), $(\bar{R}(U,V)W)^{\perp}$ denotes the normal component of $\bar{R}$. U, V, W and Z are vector fields tangent to M and N_1, N_2 are vector fields normal to M.

Definition (1.3.4). A vector sub-bundle μ of the normal bundle $T^{\perp}(M)$ is said to be *parallel* (in the normal bundle) if

$$\nabla_U^{\perp} N \in \mu$$

for any $U \in T(M)$ and any local cross section N in μ.

1.4 Some Special Submanifolds.

On an almost Hermitian manifold $\bar{M}$,

$$g(JU,JV) = g(U,V)$$

for all vector fields U, Von $\bar{M}$. In other words,

$$g(JU,U) = 0$$

i.e., $JU \perp U$ for each vector field U on $\bar{M}$. Hence for a submanifold M of $\bar{M}$ if $U \in T_p(M)$, JU may or may not belong to $T_p(M)$. Thus the action of the almost complex structure J on the tangent vectors of the submanifold of the almost Hermitian manifold gives rise to its classification into invariant and anti-invariant submanifolds. These submanifolds, are defined as follows

Definition (1.4.1) [34]. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *invariant* (or *holomorphic*) if

$$J(T_p(M)) = T_p(M)$$

for all $p \in M$.

Definition (1.4.2) [42]. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *totally real* (or *anti-holomorphic*) if

$$J(T_p(M)) \subseteq T_p^{\perp}(M)$$

for all $p \in M$.

Remark (1.4.1). Notice that M is a holomorphic submanifold of $\bar{M}$ if and only if for any non zero vector U tangent to M at any point $p \in M$, the angle between JU and the tangent space T_pM is equal to zero. Whereas M is totally real if and only if for any non zero tangent vector U tangent to M at any point $p \in M$, the angle between JU and T_pM is equal to $\pi/2$.

In 1978, A. Bejancu ([1], [2]) considered a new class of submanifolds of an almost Hermitian manifold of which the above classes namely invariant and totally real submanifolds are particular cases and named this class of submanifolds as CR-submanifold that is, a CR-submanifold provides a single setting to study the invariant and anti-invariant submanifolds of an almost Hermitian manifold.

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g, and M, a Riemannian submanifold immersed in $\bar{M}$. At each point $p \in M$, let D_p be the maximal holomorphic subspace of the tangent space $T_p(M)$ i.e.,

$$D_p = T_p(M) \cap JT_p(M).$$

If the dimension of D_p is same for all $p \in M$ we get a holomorphic distribution D on M.

Definition (1.4.3) . A Riemannian submanifold is said to be a *CR-submanifold* of an almost Hermitian manifold $\bar{M}$ if there exist on M a C^∞-holomorphic distribution D such that its orthogonal complementary distribution $D^\perp$ is totally real i.e., $JD^\perp \subseteq T_p^\perp(M)$ for all $p \in M$.

Clearly every real hypersurface M of an almost Hermitian manifold is a CR-submanifold if $\dim(M) > 1$.

Remark (1.4.2). We observe from the above definition that the dimension of D is always even and $JD^\perp$ is a sub-bundle of $T^\perp(M)$, the normal bundle splits as,

$$T^\perp M = JD^\perp \oplus \mu$$

where μ is the complement of $JD^\perp$ in $T^\perp M$ and it is easy to verify that μ is invariant under J.

Definition (1.4.4). A CR-submanifold M is said to be *proper* if neither D nor $D^\perp = 0$. Obviously if $D = 0$, then M is *totally real submanifold* and if $D^\perp = 0$, then M is a *holomorphic submanifold.*

Note. Throughout the dissertation M denotes a submanifold of the ambient space $\bar{M}$, unless mentioned otherwise.

Definition (1.4.5). A CR-submanifold is called *anti-holomorphic* submanifold if $JD_p^\perp = T_p^\perp M$ for all $p \in M$.

Definition (1.4.6). A CR-submanifold M is called a CR-product if it is locally a Riemannian product of a holomorphic submanifold M^T and a totally real submanifold $M^\perp$.

From the above definition it follows that on a CR-product submanifold, the leaves of D and $D^\perp$ are totally geodesic in M and vice-versa.

We know that the leaves of a distribution D on a manifold M are totally geodesic in M if and only if $\nabla_X Y \in D$ for all $X, Y \in D$. Thus in the setting of CR-submanifold of an almost Hermitian manifold, the leaves of D are totally geodesic in M if and only if

$$\nabla_X Y \in D \tag{1.4.1}$$

for all $X, Y \in D$. Which is equivalent to the conditions

$$\nabla_X W \in D^{\perp} \tag{1.4.2}$$

for $X \in D$ and $Z, W \in D^{\perp}$. Similarly for the totally geodesicness of the leaves of $D^{\perp}$, the conditions

$$\nabla_Z W \in D^{\perp}, \tag{1.4.3}$$

$$\nabla_Z X \in D, \tag{1.4.4}$$

for X in D and Z, W in $D^{\perp}$, are equivalent.

From the definition (1.4.6), a CR-submanifold is a CR-product if and only if the leaves of D and $D^{\perp}$ are totally geodesic in M. Hence combining (1.4.1) and (1.4.4), we conclude that a CR-submanifold of an almost Hermitian manifold is a CR-product if and only if

$$\nabla_U X \in D \tag{1.4.5}$$

for all $U \in M$. Similarly, combining (1.4.2) and (1.4.3), it is concluded that a CR-submanifold is a CR-product if and only if

$$\nabla_U Z \in D^{\perp}. \tag{1.4.6}$$

Conditions (1.4.5) and (1.4.6) are equivalent because

$$g(\nabla_U X, Z) = 0 \;\Leftrightarrow\; g(X, \nabla_U Z) = 0.$$

The generalization of Riemannian products namely warped product, doubly and twisted product manifolds are defined as follows:

Definition (1.4.7) [16]. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifold with Riemannian metric g_1 and g_2 respectively and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manidfold $M_1 \times_f M_2 = (M_1 \times M_2, g)$, where

$$g = g_1 + f^2 g_2.$$

More explicitly, if U is tangent to $M = M_1 \times_f M_2$ at (p, q), then

$$||U||^2 = ||d\pi_1 U||^2 + f^2(p)||d\pi_2 U||^2$$

where $\pi_i (i = 1, 2)$ are the canonical projections on $M_1 \times M_2$.

Note In the above definition if f be a positive differentiable function on M i.e. $M_1 \times M_2$. Then the warped product is nothing but a twisted product of M_1 and M_2.

For any vector field U tangent to M, we put

$$JU = PU + FU, \tag{1.4.7}$$

where PU and FU are the tangential and normal components of JU respectively. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued one form on TM. It is easy to see that P and F are annihilators on $D^\perp$ and D respectively. Similarly for any vector N normal to M, if we put

$$JN = tN + fN \tag{1.4.8}$$

with tN and fN as tangential and normal component of JN respectively then f can be treated as an endomorphism of the normal bundle $T^\perp M$ and t, a tangent bundle valued 1-form on $T^\perp M$ with kernel as $JD^\perp$ and μ respectively.

The covariant differentiation of the operators P, F, t and f are defined respectively as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V \quad (1.4.9)$$

$$(\bar{\nabla}_U F)V = \nabla_U^{\perp} FV - F\nabla_U V \quad (1.4.10)$$

$$(\bar{\nabla}_U t)N = \nabla_U tN - t\nabla_U^{\perp} N \quad (1.4.11)$$

$$(\bar{\nabla}_U f)N = \nabla_U^{\perp} fN - f\nabla_U^{\perp} N \quad (1.4.12).$$

Definition (1.4.8) [15]. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *generic submanifold* if the maximal holomorphic subspace

$$D_x = T_x(M) \cap JT_xM,$$

has a constant dimension for each $x \in M$ and it defines a differentiable distribution on M.

In this case the tangent space T_xM of M at each point $x \in M$ is decompose as

$$T_xM = D_x \oplus D_x^{\perp}$$

here $D_x^{\perp}$ is the orthogonal complement of the holomorphic subspace D_x and is not necessarily totally real as was in the case of $CR-$submanifold. For this reason generic submanifold is a generalized version of CR-submanifold. The distribution $D^{\perp}$ on a generic submanifold is known as *purely real distribution*.

Now, in view of the remark (1.4.1), we have a third important class of submanifold of An almost Hermitian manifold (in particular of a Kaehler manifold), called slant submanifolds.

A *slant submanifold* is defined as a submanifold of $\bar{M}$ such that for any non zero vector $U \in T_x(M)$, the angle $\theta(U)$ between JU and the tangent space $T_x(M)$ is constant (which is independent of the choice of the point $x \in$

M and the choice of the tangent vector $U \in T_x(M)$). It is obvious to realize that holomorphic and totally real submanifolds are special classes of slant submanifolds. A slant submanifold is called *proper* if it is neither holomorphic submanifold nor totally real submanifold.For a slant submanifold, we have

$$g(PX, PY) = cos^2\theta g(X, Y) \tag{1.4.13}$$

$$g(FX, FY) = sin^2\theta g(X, Y) \tag{1.4.14}$$

for X, Y tangent to M. A natural generalization of CR-submanifolds in terms of slant distribution was given by N. Papaghuic [ ]. These submanifolds are known as semi-slant submanifold. He defined these submanifolds as

Definition (1.4.9). A submanifold M of an almost Hermitian manifold is called a semi-slant submanifold if it is endowed with two orthogonal complementary distributions D and D^0 such that D is holomorphic and D^0 is slant i.e., the angle $\theta(X)$ between JX and D^0_x is constant for each $X \in D^0_x$.

Hence, CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \pi/2$ and $D = \{0\}$ respectively.

CHAPTER II

CR-PRODUCT SUBMANIFOLDS OF ALMOST HERMITIAN MANIFOLDS

2.1 Introduction.

CR-submanifolds have been extensively studied by A. Bejancu, B.Y. Chen, D.E. Blair, M. Sato, K. Sekigawa and others [1], [2], [7], [11], [12], [38]. One of the interesting class of CR-submanifolds is the class of CR-product submanifolds . A CR-submanifold is called a CR-product if it is locally a Remannian product of the leaves of the distributions D and $D^{\perp}$. The purpose of this chapter is to present results concerning the necessary and sufficient conditions for the CR-submanifolds to become CR-product submanifolds when the ambient manifold is almost Hermitian.

Let M be a CR-submanifold of an almost Hermitian manifold $(\overline{M}, J, g)$ with holomorphic and totally real distributions D and $D^{\perp}$ respectively.We shall denote by U, V as vectors tangential to M, X,Y as vectors in the holomorphic distribution D and Z, W in the totally real distribution $D^{\perp}$. For any vector feild U tangent to M, we put

$$JU = PU + FU, \tag{2.1.1}$$

where PU and FU are the tangential and normal components of JU respectively. Hence P is an endomorphism of the tangent bundle TM and F is a normal valued 1-form on TM. We shall denote by B and C the projection morphism of TM onto D and $D^{\perp}$ respectively.

Frobenius theorem guarentees the exsistence of leaves of D and $D^{\perp}$ if the distributions are integrable. A. Bejancu [4] obtained the following integrability conditions for the distributions D and $D^{\perp}$ on a CR-submanifold of an almost Hermitian manifold $\overline{M}$.

The Nijenhuis tensors of a(1-1) tensor A is given by

$$S_A(X,Y) = A[X,AY]+A[AX,Y]-A^2[X,Y]-[AX,AY]. \quad (2.1.2)$$

Then by using (2.1.1) and (2.1.2), we obtain

$$S(X,Y) = C([X,Y])+F([AX,Y]+[X,AY])-S_P(X,Y), \quad (2.1.3)$$

for any X, Y in D.

2.2 CR-product Submanifolds of an Almost Hermitian Manifolds.

We shall first give the conditions for the integrability of the distributions proved by A. Bejancu [4], K.A. Khan, V.A. Khan and S.I Husain [26]. From the above equations we have:

Proposition 2.2.1 [4]. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution D is integrable if and only if

$$[S(X,Y)]^T = -S_p(X,Y), \quad (2.2.1)$$

for any X, Y in D,where T denotes the tangential part.

Taking the normal part in (2.1.3) we obtain

$$[S(X,Y)]^{\perp} = -F([PX,Y] + [X,PY]), \quad (2.2.2)$$

for any X, Y in D.

Proposition 2.2.2 [4]. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$.Then the distribution D is integrable if and only if

$$[S(X,Y)]^T = 0, \quad (2.2.3)$$

$$CS_p(X, Y) = 0, \tag{2.2.4}$$

for any X, Y in D.

Proof. Suppose D is integrable. Then (2.2.3) follows from (2.2.2). It is easy to observe that

$$PX = JBX \quad \text{which implies} \quad P^2 = -B.$$

Using (2.1.1), we obtain

$$S_p(X, Y) = P[X, PY] + P[PX, Y] + B([X, Y] - [PX, PY],$$

for any X, Y in D. Since D is integrable, we have

$$S_p(X, Y) \in D,$$

which is equivalent to (2.2.4).

Conversely, suppose (2.2.3) and (2.2.4) are satisfied. Then from (2.2.2) and (2.2.3) we have

$$C([JX, Y] + [X, JY]) = 0,$$

which implies that $\quad C([X, Y] - [JX, JY]) = 0.$

Hence, $\quad C(S(X, Y)^T) = 0.$

On the other hand, from (2.1.3) we obtain

$$C(S(X, Y)^T) = C([X, Y]) - C(S_p(X, Y)).$$

Thus by (2.2.4) we obtain$C([X, Y]) = 0$,

that is, D is integrable, which proves the theorem completely.

Now, if we take Z, W in $D^{\perp}$ then we get

$$S_p(Z, W) = B([Z, W]). \tag{2.2.5}$$

Thus for integrability of the totally real distribution $D^{\perp}$ we have

Proposition 2.2.3 [4]. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if the Nijenhuis tensor of P vanishes identically on $D^{\perp}$.

K.A. Khan, V.A. Khan and S.I. Husain [26] considered the tensors $\mathcal{P}$ and Q to obtain integrability conditions of the canonical distributions D and $D^{\perp}$ on a CR-submanifold of an almost Harmitian manifold. They also studied the geometric properties of the leaves of the distributions using these tensors.

In the following paragraphs, we introduce $\mathcal{P}$ and Q and establish some of their important properties for the later use.

Let $\overline{M}$ be an almost Hermitian manifold and M be a CR-submanifold of $\overline{M}$. Then for U, V in TM, we have

$$(\overline{\nabla}_U J)V = \overline{\nabla}_U JV - J\overline{\nabla}_U V.$$

Making use of (1.3.2) and (1.4.10), the above equation takes the form

$$(\overline{\nabla}_U J)V = (\nabla_U P)V - A_{FV}U - th(U,V) + (\nabla_U F)V + h(U, PV) - fh(U,V).$$

Denoting the tangential and normal parts of $(\overline{\nabla}_U J)V)$ in the above equation by $\mathcal{P}_U V$ and $Q_U V$ respectively, we can write

$$\mathcal{P}_U V = (\nabla_U P)V - A_{FV}U - th(U,V), \tag{2.2.6}$$

$$Q_U V = (\nabla_U F)V + h(U, PV) - fh(U,V). \tag{2.2.7}$$

Similarly, for $N \in TM^{\perp}$ denoting by $\mathcal{P}_U N$ and $Q_U N$ respectively the tangential and normal parts of $(\overline{\nabla}_U J)$, we find that

$$\mathcal{P}_U N = (\nabla_U t)N + PA_N U - A_{fN} U, \tag{2.2.8}$$

$$Q_U N = (\nabla_U f)N + h(tN, U) + fA_N U.$$

The following properties of $\mathcal{P}$and Q can be verified through straight forward computations.

$p1.$ (i) $\mathcal{P}_{U+V} W = \mathcal{P}_U W + \mathcal{P}_V W;$ (ii) $Q_{U+V} W = Q_U W + Q_V W,$

$p2.$ (i) $\mathcal{P}_U (V+W) = \mathcal{P}_U V + \mathcal{P}_U W;$ (ii) $Q_U (V+W) = Q_U V + Q_U W,$

$p3.$ $g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W),$

$p4.$ $g(Q_U V, N) = -g(V, \mathcal{P}_U N),$

$p5.$ $\mathcal{P}_U JV + Q_U JV = -J(\mathcal{P}_U V + Q_U V),$

for all U, V and W in TM and N in $TM^{\perp}$.

Proposition 2.2.4 [26]. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution D is integrable if and only if

$$Q_X Y - Q_Y X = h(X, JY) - h(JX, Y), \tag{2.2.9}$$

for each X, Y in D.

Proof. For $N \in TM^{\perp}$ we have

$$g(\overline{\nabla}_X JY - \overline{\nabla}_Y JX, N) = g(h(X, JY) - h(JX, Y), N),$$

$$\text{or, } g(J(\overline{\nabla}_X Y - \overline{\nabla}_Y X) + Q_X Y - Q_Y X, N) = g(h(X, JY) - h(JX, Y), N),$$

$$\text{or, } g(F[X, Y], N) = g(h(X, JY) - h(Y, JX) + Q_Y X - Q_X Y, N).$$

It follows from the above equality that the distribution D is integrable if and only if

$$g(h(X, JY) - h(JX, Y) + Q_Y X - Q_X Y, N) = 0,$$

for each X, $Y \in$ D and $N \in TM^{\perp}$. This proves the assertion.

For the integrability of the totally real distribution, we have,

Proposition 2.2.5 [26]. Let M be a CR-submanifold of an almost Hermitian manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if

$$\mathcal{P}_Z W - \mathcal{P}_W Z = A_{JZ} W - A_{JW} Z, \tag{2.2.10}$$

for each Z, W in $D^{\perp}$.

Proof. For $U \in TM$, we may write

$$g(J[Z, W], U) = g(J\overline{\nabla}_Z W - J\overline{\nabla}_W Z, U)$$

$$= g(\overline{\nabla}_Z JW - \overline{\nabla}_W JZ - \mathcal{P}_Z W + \mathcal{P}_W Z, U).$$

This shows that $D^{\perp}$ is integrable if only if

$$\mathcal{P}_Z W - \mathcal{P}_W Z = A_{JZ} W - A_{JW} Z.$$

This completes the proof.

It may be seen through direct calculation that

$$g(Q_X Y - Q_Y X + h(JX, Y) - h(X, JY), \xi) = 0,$$

for all X, $Y \in$D and ξ in μ, and

$$g(\mathcal{P}_Z W - \mathcal{P}_W Z + A_{JW} Z - A_{JZ} W, Z') = 0,$$

for all W, Z and Z' in $D^{\perp}$. Hence it follows that the condition (2.2.9) is satisfied if and only if

$$g(Q_XY - Q_YX + h(JX,Y) - h(X,JY), JZ) = 0, \tag{2.2.11}$$

and the condition (2.2.10) is satisfied if and only if

$$g((\mathcal{P}_ZW - \mathcal{P}_WZ + A_{JW}Z - A_{JZ}W, X) = 0. \tag{2.2.12}$$

Now when D and $D^{\perp}$ are integrable, the existence of the leaves is guarenteed by Frobenius theorem. Regarding these leaves of the distribution, we establish:

Proposition 2.2.6 [26]. The leaves of the holomorphic distribution D on a CR-submanifold of an almost Hermition manifold $\overline{M}$ are totally geodesic in M if and only if either of the following equivalent conditions hold.

(i). $\mathcal{P}_XY + th(X,Y) \in D$,

(ii). $\mathcal{P}_XZ + A_{JZ}X \in D^{\perp}$,

(iii). $Q_XY - h(X,JY) \in \mu$,

for each X, Y in D and Z in $D^{\perp}$.

Proof. The proof of the first part follows directly from (2.2.6) whereas part (i) and (ii) are equivalent in view of property ($p3$). For part (iii) we observe that

$$g(\mathcal{P}_XY + th(X,Y), Z) = 0.$$

As $Q_XY \in T^{\perp}$M and $g(th(X,Y), Z) = g(Jh(X,Y), Z)$, the above eqation may be written as:

$$g(\mathcal{P}_XY + Q_XY + Jh(X,Y), Z) = 0,$$

or, $g(J(\mathcal{P}_X Y + Q_X Y - h(X,Y), JZ) = 0.$

Now taking account of the fact that $\mathcal{P}_X Y \in TM$ and using the property $(p5)$, the above equation gives

$$g(-Q_X JY - h(X,Y), JZ) = 0.$$

Replacing Y by JY, we obtain

$$g(Q_X Y - h(X, JY), JZ) = 0,$$

i.e, $Q_X Y - h(X, JY) \in \mu.$

This shows that (i) and (ii) are equivalent. Hence the Proposition is proved completely.

Regarding the leaves of $D^{\perp}$, we have

Proposition 2.2.7 [26]. The leaves of the totally real distribution $D^{\perp}$, on a CR-submanifold M of an almost Hermition manifold $\overline{M}$ are totally geodesic in M if and only if any of the following equivalent conditions hold

(i). $\mathcal{P}_Z X + A_{JZ} W \in D^{\perp}$,

(ii). $\mathcal{P}_Z X + th(X,Z) \in D$,

(iii). $Q_W X - h(W, JX) \in \mu$,

for all X, Y in D and Z in $D^{\perp}$.

Proof. By the argument similar to the previous Proposition (i) follows from (2.2.6) and part (ii) and (iii) are equivalent to (i) in view of property $(p3)$ and $(p4)$ of tensor $\mathcal{P}$ and Q.

Recalling, that a CR-submanifold with leaves of both the distribution D and $D^{\perp}$ being totally geodesic in M, the following can be easily established by making use of the Proposition 2.2.6 and 2.2.7 and the properties of $\mathcal{P}$ and Q.

Theorem 2.2.1 [26]. For a CR-submanifold of an almost Hermition manifold $\overline{M}$, the following are equivalent,

(i) M is a CR-product,

(ii) $\mathcal{P}_U X + th(U, X) \in D$.

(iii) $\mathcal{P}_U Z + A_{JZ}U \in D^{\perp}$,

(iv) $Q_U X - h(U, JX) \in \mu$,

for each X in D, Z in $D^{\perp}$ and U in TM.

Now we shall give results regarding integrability and totally geodesicness of the distributions D and $D^{\perp}$ when the ambient manifold is Kaheler i.e. when $(\overline{\nabla}_U J) = 0$ for all U in TM.

2.3 CR-product Submanifolds of Kaehler Manifolds.

The integrability conditions of D and $D^{\perp}$ in case of $\overline{M}$ is Kaehler were obtained by A. Bejancu [4], B.Y.Chen and D.E. Blair [7]. For the integrability of the distribution D we have

Lemma 2.3.1 [11]. Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$. Then the holomorphic distribution D is integrable if and only if

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ), \tag{2.3.1}$$

for any X, Y in D and Z in $D^{\perp}$.

Proof. For a Kaehler manifold $\overline{M}$, we have $\overline{\nabla}J = 0$. If M is a CR-submanifold in $\overline{M}$, Gauss and Weingarten's formula imply

$$J\nabla_U Z + Jh(U, Z) = -A_{JZ}U + \nabla_U^{\perp} JZ,$$

for U in TM and Z in $D^{\perp}$. Taking the innerproduct with JY in both sides of above equation we get,

$$g(J\nabla_U Z, JY) = g(A_{JZ}U, JY),$$

now using the relation between shape operator A and second fundamental form h, we

get
$$g(J\nabla_U Z, JY) = -g(h(U, JY), JZ),$$

for U in TM, Y in D and Z in $D^{\perp}$. The above equation can be written as

$$g(Z, \nabla_U Y) = g(h(U, JY), JZ),$$

from this we have for all X, Y in D and Z in $D^{\perp}$

$$g(Z, [X, Y]) = g(h(X, JY) - h(JX, Y), JZ).$$

It follows from the equality that the distribution D is integrable if and only if

$$g(h(X, JY), JZ) = g(h(JX, Y), JZ),$$

for each X, Y in D and Z in $D^{\perp}$. This prove the assertion.

Recalling that the leaves of an integrable distribution H (say) on a manifold M are totally geodesic if and only if $\nabla_U V \in H$ for all U, V in H.

Now we prove the condition of the totally geodesicness of the leaves of D

Lemma 2.3.2 [11]. The leaves of the holomorphic distribution D on a CR-submanifdold M of a Kaehler manifold $\overline{M}$ are totally geodesic in M if and only if

$$g(h(D,D), JD^{\perp}) = 0. \tag{2.3.2}$$

Proof. For X, Y in D, Z in $D^{\perp}$,

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\overline{\nabla}_X Y, Z) \\ &= g(\overline{\nabla}_X JY, JZ) \end{aligned}$$

Therefore,

$$g(\overline{\nabla}_X Y, Z) = h(X, JY), JZ). \tag{2.3.3}$$

If (2.3.2) is satisfied then by the lemma 2.3.1 D is integrable. Further from the (2.3.3) its leaves are totally geodesic in M.

Coversely if D integrable, then in view of (2.3.3) and (2.3.2) holds.

For the integrability of the distribution $D^{\perp}$, we need the following lemma:

Lemma 2.3.3 [11]. Let M be a CR-submanifold of a Kahler manifold $\overline{M}$. Then

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X), \tag{2.3.4}$$

$$A_{JZ}W = A_{JW}Z, \tag{2.3.5}$$

for all X in D, Z, W in $D^{\perp}$.

Proof. Let $\overline{M}$ be a Kaehler manifold. Then we have $\overline{\nabla}J = 0$. If M is a CR-submanifold of $\overline{M}$, then by using Gauss and Weingarten formula we have

$$J\nabla_U Z + Jh(U, Z) = -A_{JZ}U + \nabla^{\perp}_U JZ.$$

Taking the innerproduct with JX

$$g(J\nabla_U Z, JX) = g(-A_{JZ}U, JX),$$

$$g(\nabla_U Z, X) = g(JA_{JZ}U, X).$$

For equation (2.3.5), we have

$$\begin{aligned} g(A_{JZ}W,U) &= g(h(U,W),JZ) \\ &= -g(Jh(U,W),Z) \\ &= -g(J(\overline{\nabla}_U W - \nabla_U W),Z) \\ &= -g(J\overline{\nabla}_U W - J\nabla_U W, Z) \\ &= -g(\overline{\nabla}_U JW, Z)+g(J\nabla_U W,Z) \\ &= -g(-A_{JW}U, Z) - g(\nabla^{\perp}_U JW, Z) - g(\nabla_U W, JZ). \end{aligned}$$

The last two terms in the right hand side vanishes as the corresponding vector fields are perpendicular. Thus we get

$$g(A_{JZ}W, U) = g(A_{JW}Z, U),$$

for all $U \in TM$. This verifies equation (2.3.5).

Lemma(2.3.4) [11]. Let M be a CR-submanifold of a Kaehler manifold $\overline{M}$. Then for any $Z, W \in D^{\perp}$ we have

$$\nabla^{\perp}_W JZ - \nabla^{\perp}_Z JW \in JD^{\perp}. \tag{2.3.6}$$

Proof. For $Z \in D^{\perp}$ and $\xi \in \mu$, by Weingarten formula, we have

$$\overline{\nabla}_Z J\xi = -A_{J\xi}Z + \nabla^{\perp}_Z J\xi,$$

which implies $\quad g(\overline{\nabla}_Z J\xi, W) = g(-A_{J\xi}Z, W).$

$$
\begin{aligned}
g(A_{J\xi}Z, W) &= -g(J\overline{\nabla}_Z\xi, W) \\
&= g(\overline{\nabla}_Z\xi, JW) \\
&= -g(\xi, \overline{\nabla}_Z JW) \\
&= -g(\xi, \nabla^{\perp}_Z JW).
\end{aligned}
$$

Thus we obtain

$$g(\xi, \nabla^{\perp}_W JZ - \nabla^{\perp}_Z JW) = g(A_{J\xi}Z, W) - g(A_{J\xi}W, Z) = 0.$$

Since this is true for all $Z, W \in D$ and $\xi \in \mu$, where μ denotes the invariant part of the normal bundle, equation (2.3.6) holds.

Now, for $Z, W \in D^{\perp}$, consider $[Z, W]$

$$
\begin{aligned}
J[J, W] &= J(\nabla_Z W - \nabla_W Z) \\
&= J(\overline{\nabla}_Z W - \overline{\nabla}_W Z) \\
&= \overline{\nabla}_Z JW - \overline{\nabla}_W JZ \\
&= -A_{JW}Z + \nabla^{\perp}_Z JW + A_{JZ}W - \nabla^{\perp}_W JZ \\
&= (A_{JZ}W - A_{JW}Z) + (\nabla^{\perp}_Z JW - \nabla^{\perp}_W JZ).
\end{aligned}
$$

The first bracket in the right hand side of the above equation is zero by virtue of (2.3.5) whereas the vector field in the second bracket on the right hand side of the obove equation belongs to $JD^{\perp}$ in view of the lemma 2.3.4. Thus, we have

$$
\begin{aligned}
&J[Z, W] \in JD^{\perp}, \\
\text{or,} \quad &[Z, W] \in D^{\perp},
\end{aligned}
$$

for all $Z, W \in D^{\perp}$. This proves

Lemma 2.3.5 [11]. The totally real distribution $D^{\perp}$ of any CR-submanifold in a Kaehler manifold is integrable.

For the totally geodesicness of leaves of $D^\perp$ we have

Lemma 2.3.6 [11]. For a CR-submanifold M in a Kaehler manifold leaves of $D^\perp$ are totally geodesic in M if and only if

$$g(h(D, D^\perp), JD^\perp) = 0.$$

Proof. If the leaves of $D^\perp$ are totally geodesic in M thus by definition

$$\nabla_Z W \in D^\perp,$$

for all $Z, W \in D^\perp$

$$g(\nabla_Z W, X) = 0,$$

for all $X \in D$ and $Z, W \in D^\perp$

$$g(W, \nabla_Z X) = g(\overline{\nabla}_Z X, W) = g(\overline{\nabla}_Z JX, JW) = g(h(JX, Z), JW) = 0.$$

$$\Leftrightarrow g(h(D, D^\perp), JD^\perp) = 0.$$

Thus making use of Lemma 2.3.2 and 2.3.6 it follows that a CR-submanifold of a Kaehler manifold is a CR-product if and only if

$$A_{JD^\perp} D = 0. \tag{2.3.7}$$

In terms of endomorphism P the condition for CR-product submanifolds is as follows:

Theorem 2.3.1 [11]. A CR-submanifold M of a Kaehler manifold $\overline{M}$ is a CR-product if and only if P is parallel i.e.,

$$\overline{\nabla} P = 0.$$

Proof. As $\overline{M}$ is Kaehler,

$$\overline{\nabla}_U JV = J\overline{\nabla}_U V,$$

for any vectors U, V tangent to M. On using (2.1.1) and Gauss formula, the above equation becomes,

$$\overline{\nabla}_U(PU + FU) = J(\nabla_U V + h(U, V)),$$

which on applying Gauss and Weingarten formula again gives

$$\begin{aligned} & \nabla_U PV + h(U, PV) - A_{FV}U + D_U FV \\ = \;& P\nabla_U V + F\nabla_U + th(U, V) + fh(U, V). \end{aligned}$$

Now compairing tangential parts in both sides of the above equation and using

$$(\overline{\nabla}_U P)V = \nabla_U PV - P\nabla_U V,$$

we get

$$(\overline{\nabla}_U P)V = th(U, V) + A_{FV}U.$$

If P is parallel then from the above equation

$$th(U, V) = -A_{FV}U,$$

for any vectors U, V tangent to M. In particular if X in D then $FX = 0$. Hence above equation implies

$$\begin{aligned} & th(U, X) = 0, \\ \text{i.e.,} \quad & A_{JZ}X = 0, \end{aligned}$$

for any Z in $D^\perp$ and X in D. Thus by Lemma 2.3.1 and 2.3.2 D is integrable and its leaves are totally geodesic in M Similarly on using Lemma 2.5.6 leaves of $D^\perp$ are totally geodesic in M. Thus M is a CR-product.

Conversely, if M is a CR-product, then

$$g(\nabla_X Y, Z) = 0 \quad \text{and} \quad g(\nabla_Z W, Y) = 0,$$

for all X, Y in D and Z, W in $D^{\perp}$

$$\nabla_X Y \in D \quad \text{and} \quad \nabla_Z Y \in D,$$

$$\text{i.e.,} \qquad \nabla_U Y \in D,$$

for all U in TM, Y in D. On using the fact that M is a CR-product and Gauss formula we obtain

$$Jh(U,V) = h(U,JV).$$

Therefore from the equation

$$(\overline{\nabla}_U P)V = th(U,Y) + A_{FY}U,$$

we get
$$(\overline{\nabla}_U P)V = 0.$$

Similarly, as $\nabla_U Z \in D^{\perp}$, for any Z in $D^{\perp}$ and U tangent to M, it is easy to see that

$$\overline{\nabla}_U Z = 0.$$

Thus if M is a CR-product, then

$$\overline{\nabla}_U P = 0.$$

This proves the theorem completely.

CHAPTER III

CR-PRODUCTS SUBMANIFOLDS OF NEARLY KAEHLER AND ALMOST KAEHLER MANIFOLDS

3.1 Introduction.

In the previous chapter we have given the results concerning the necessary and sufficient conditions for the CR-submanifolds to become CR-product submanifolds when the ambient manifold is almost Hermition and in particular Kaehler manifold.

A nearly Kaehler and almost Kaehler structures on an almost Hermition manifold are given by $(\overline{\nabla}_X J)X = 0$ and $(\overline{\nabla}_X J)X + (\overline{\nabla}_{JX} J)JX = 0$ respectively. Obviously these are weaker conditions than Kaehler condition of $(\overline{\nabla}_X J) = 0$ for all X in TM. In this chapter we shall discuss and prove the results concerning CR-product submanifolds when the ambient manifold is nearly Kaehler and almost Kaehler.

A. Bejancu, Gray , Sekigawa, K.A.Khan *et al.* have extensively studied CR-submanifolds of these manifolds([23],[27], etc).

3.2 CR-product Submanifolds of Nearly Kaehler Manifolds.

Nearly Kaehler manifolds are an important and very interesting class of almost Hermition manifolds. An exciting example is that of S^6, which is a nearly Kaehler manifold and admits no proper CR-propduct submanifolds [37] First we shall prove the integrability conditions for the distributions D and $D^\perp$ on a CR-submanifold of a nearly Kaehler manifold given by Sato [38] and Urbano [40].

Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then, by using the formulas of Gauss, Weingarten and (1.2.10) we obtain

$$[JX, Y] + [X, JY] = \frac{1}{2}J(S(X, Y) - J([X, Y])$$

$$+\nabla_{JX}Y\nabla_{JY}X + h(JX, Y) - h(X, Y), \tag{3.2.1}$$

for any X, Y in D, where ∇ is the induced Reimannian connection on M and h denotes the second fundamental form of the immersion of M into $\overline{M}$. Taking into account that ∇ is a torsion free connection, from (3.2.1) we get

$$h(X, JY) - h(JX, Y) = \frac{1}{2}J(S(X, Y)) + J([X, Y])$$

$$+\nabla_Y JX - \nabla_X JY, \tag{3.2.2}$$

for any X, Y in D.

Proposition 3.2.1 [38]. Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then the distribution D is integrable if and only if the following conditions are satisfied

$$h(X, JY) = h(JX, Y), \tag{3.2.3}$$

$$S(X, Y) \in D, \tag{3.2.4}$$

for any X, Y in D.

Proof. Suppose D is integrable, then by (3.2.1)

$$h(X, JY) - h(JX, Y) = \frac{1}{2}J(S(X, Y)). \tag{3.2.5}$$

Using (2.2.1), (2.2.3), and (2.2.4) and the above equation, we get (3.2.3). Also, from (2.2.1), (2.2.3) and (2.2.4) it follows that (3.2.4) holds.

Conversely, suppose (3.2.3) and (3.2.4) are satisfied, then by using (3.2.2) we obtain

$$J([X,Y]) = \nabla_X JY - \nabla_Y JX - \frac{1}{2}J(S(X,Y)). \tag{3.2.6}$$

We note that for each Z in $D^\perp$ there exist V in $TM^\perp$ such that $Z = JV$ then using (3.2.4) and (3.2.6) we obtain

$$g([X,Y],JV) = -g(J[X,Y],V) = 0.$$

Hence [X,Y]$\in D$ for each X, Y in D, that is, D is integrable .

It is easy to see that Nijenhuis tensor S for the nearly Kaehler manifold $\overline{M}$ takes the following form

$$S(X,Y) = -4J(\overline{\nabla}_Y J)X. \tag{3.2.7}$$

Using (3.2.7) and the Proposition 3.2.1 gives the condition of integrability of the distribution D as follows:

Proposition 3.2.2 [40]. Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then the distribution D integrable if and only if

$$(\overline{\nabla}_X J)Y \in D,$$

$$\text{and} \quad h(X,JY) = h(JX,Y),$$

for any X, Y in D.

Also, from Proposition 3.2.2 and (3.2.7), we obtain

Corollary 3.2.1 . Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then the distribution D is integrable if and only if

$$J(X,JY) = h(JX,Y) \quad \text{and} \quad S(X,Z)^\perp \in D^\perp,$$

for any X in D and Z in $D^{\perp}$.

From (3.2.2) it follows that

$$g(h(X, JY) - h(Y, JX), \xi) = g(S(X, Y), J\xi) = 0, \qquad (3.2.8)$$

for any X, Y in D and ξ in μ. Thus from equation (3.2.8), we have

Proposition 3.2.3 . The condition (3.2.3) is satisfied if and only if

$$g(h(X, JY) - h(Y, JX), JZ) = 0,$$

for any X, Y in D and Z in $D^{\perp}$.

Now in the following we will be concerned with the integrability of $D^{\perp}$ on a CR-submanifold of a nearly Kaehler manifold.

Proposition 3.2.4 [40]. Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then the distribution $D^{\perp}$ is integrable if and only if

$$g((\overline{\nabla}_Z J)W, X) = 0, \qquad (3.2.9)$$

for any Z, W in $D^{\perp}$ and X in D.

Proof. For all X, Z, W in $\overline{M}$, we have

$$3d\Omega(Z, W, X) = g((\overline{\nabla}_Z)W, X) + g((\overline{\nabla}_W)X, Z) + g((\overline{\nabla}_X)W, Z).$$

Using (1.2.10), we get

$$3d\Omega(Z, W, X) = g((\overline{\nabla}_Z)W, X), \qquad (3.2.10)$$

for any Z, W in $D^{\perp}$ and X in D. On the other hand, by direct computation we get

$$3d\Omega(Z, W, X) = g([Z, W], JX). \qquad (3.2.11)$$

Thus from (3.2.10) and (3.2.11), we obtain (3.2.9) and the Proposition is proved.

Taking account of (1.2.10), the Gauss formula and from Proposotion 3.2.4 we obtain:

Corollary 3.2.2 [38]. Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable if and only if

$$g(h(Z,X),JW) = g(h(W,X),JZ),$$

for any Z, W in $D^{\perp}$ and X in D.

From the above we can also see that:

Proposition 3.2.5 . Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. If $D^{\perp}$ is integrable then the leaves of $D^{\perp}$ are totally geodesic in M if and only if

$$g(h(Z,X),JW) = 0,$$

for any Z, W in $D^{\perp}$ and X in D.

Now by using (3.2.7) and Proposition 3.2.4 we obtain:

Corollary 3.2.3 . Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then $D^{\perp}$ is integrable if and only if

$$[S(X,Z)]^{T} \in D,$$

for any X in D and Z in $D^{\perp}$.

Combining the above results, we get:

Proposition 3.2.6 . Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$. Then both the distributions D and $D^{\perp}$ are integrable if and only if

$$h(X, JY) = h(JX, Y), \quad \text{and} \quad S(X, Z)^{T} = 0,$$

for any X in D and Z in $D^{\perp}$.

Now we shall obtain the conditions for integrability of these distributions on a CR-submanifold of a nearly Kaehler manifold in term of the tensors $\mathcal{P}$ and Q.

In view of the relation,

$$(\overline{\nabla}_U J)V = \mathcal{P}_U V + Q_U V,$$

The nearly Kaehler condition yields

$$\mathcal{P}_U V = -\mathcal{P}_V U,$$

$$Q_U V = -Q_V U,$$

for all vector fields U, V tangent to M and therefore by Proposition 2.2.4 the necessary and sufficient conditions for the holomorphic distribution D on a CR-submanifold of a nearly Kaehler manifold, to be integrable can be written as:

$$2Q_X Y = h(X, JY) - h(JX, Y), \tag{3.2.12}$$

for all X, Y in D. Furthermore equation (3.2.7) and property ($p5$) give

$$S(X, Y)^{\perp} = 4Q_X JY. \tag{3.2.13}$$

In view of (2.1.2) and (3.2.13) the left hand side of the above equation becomes

$$F([PX, Y] + [X, PY]).$$

Hence, from (3.2.12) and (3.2.13), it follows that the holomorphic distribution D on a CR-submanifold of a nearly Kaehler manifold $\overline{M}$ is integrable if and only if

$$Q_X Y = 0, \tag{3.2.14}$$

and

$$h(X, JY) = h(JX, Y). \tag{3.2.15}$$

For the integrability of $D^\perp$, by (2.2.12) and (1.2.10), it can be seen that $D^\perp$ is integrable if and only if , for all Z, W in $D^\perp$ and X in D,

$$2g(\mathcal{P}_Z W, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X),$$

$$i.e., \quad 2g((\overline{\nabla}_Z J)W, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Now, as in nearly Kaehler manifolds

$$d\Omega(U, V, W) = 3g((\overline{\nabla}_U J)V, W),$$

the above equation yields

$$\frac{2}{3}d\Omega(Z, W, X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Further as

$$\Omega(D, D^\perp) = \Omega(D^\perp, D^\perp) = 0,$$

we get

$$\frac{2}{3}g([Z, W], X) = g(A_{JZ}W, X) - g(A_{JW}Z, X).$$

Hence, we conclude that the totally real distribution $D^\perp$ on a CR-submanifold is integrable if and only if

$$g(\mathcal{P}_Z W, X) = 0, \tag{3.2.16}$$

$$\text{or,}\quad g(A_{JZ}W, X) = g(A_{JW}Z, X), \tag{3.2.17}$$

for all Z, W in $D^{\perp}$ and X in D.

These integrability conditions lead to the following characterization:

Theorem 3.2.1 [26]. Let M be a CR-submanifold of a nearly Kaehler manifold $\overline{M}$ and suppose both the distributions D and $D^{\perp}$ are integrable then M is a CR-product if and only if

$$A_{JD^{\perp}}D = 0.$$

Proof. Making use of (3.2.14) and (3.2.15) and Proposition 2.2.6, it follows that *the leaves of the holomorphic distribution are totally geodesic in M if and only if*

$$h(X, Y) \in \mu, \tag{3.2.18}$$

for all X, Y in D.

Similarly it follows from (3.2.16) (3.2.17) and Proposition 2.2.7 that leaves of $D^{\perp}$ are totally geodesic in M if and only if

$$h(X, Z) \in \mu, \tag{3.2.19}$$

for all X in D and Z in $D^{\perp}$.

The assertion follows immediately on combining (3.2.18) and (3.2.19).

Remark. It may be noted that this condition is same as the condition obtained by B.Y.Chen [11] for the characterization of CR-product in a Kaehler manifold.

Now we shall find the conditions for the CR-product submanifolds in an almost Kaehler manifold.

3.3 CR-product Submanifolds of Almost Kaehler Manifolds.

Throughout this section we shall denote by $\overline{M}$ an almost Kaehler manifold equipped with an almost complex structure J and Hermition metric g. An almost Kaehler manifold is an almost Hermition manifold with fundamental 2-form Ω such that $d\Omega = 0$ i.e., Ω is closed. Hence on $\overline{M}$, we have

$$\Omega(U, V) = g(JU, V).$$

Let M be submanifold of $\overline{M}$, we shall denote by Ω_M, a restriction of the fundamental 2-form on M, then the $ker(\Omega_M)$ is the set of all vector fields X in TM, such that $\Omega(X, Y) = 0$ for each Y in TM. Now if $ker(\Omega_M)$ has constant rank over M, then it defines a distribution $D^{\perp}$ which is totally real with respect to the almost complex structure J.
The integrability of the totally real distribution $D^{\perp}$ in this case follows immediately from the following theorem.

Theorem 3.3.1 [30]. If ϕ is a p-form on a differentiable manifold M such that $\phi = \phi \cap \alpha$ where α is a Pfaffian form then the distribution generated by the set of all sections of ker(ϕ) is completely integrable.

We begin this section by a proposition unfolding the relations between Nijenhuis tensor $S(X, Y)$ and the vectors $\mathcal{P}_U V$ and $Q_U V$ when M is taken to be a CR-submanifold of an almost Kaehler manifold.

Proposition 3.3.1 . Let M be CR-submanifold of $\overline{M}$. Then for any U, V and W in TM and in $TM^{\perp}$.

$$2g(\mathcal{P}_U V, W) = g(U, JS(V, W))$$

$$\text{and} \qquad 2g(Q_U V, W) = g(U, JS(V, N)).$$

Proof. Using the fact that $d\Omega = 0$ on an almost Kaehler manifold and $\mathcal{P}_U V$ and $Q_U V$ are tangential normal parts of $(\overline{\nabla}_U J)V$, respectively, an easy calculation using the following formula of Gray [25]

$$2g((\overline{\nabla}_U J)V, W) = d\Omega(U, V, W) - d\Omega(U, JV, JW) - d\Omega(U, S(V, J$$

and (1.2.7), immediately yields the required relations.

By applying the above relation, the integrability conditions for the holomorphic distribution on a CR-submanifold turned out to be as follows:

Proposition 3.3.2 . The holomorphic distribution D on a CR-submanifold M of an almost Kaehler manifold $\overline{M}$ is integrable if and only if

$$g(2A_{JZ}Y - JS(Y, Z), JX) = g(2A_{JZ}X - JS(X, Z), JY),$$

for each X, Y in D and Z in $D^{\perp}$.

Proposition 3.3.3 . The leaves of the holomorphic distribution D on a CR-submanifold of $\overline{M}$ are totally geodesic in M if and only if

$$g(2A_{JZ}Y - JS(Y, Z), X) = 0.$$

for all X,Y in D and Z in $D^{\perp}$.

Proof. Applying condition (i) of Proposition 2.2.6, we get

$$g(\mathcal{P}_X Y + th(X, Y), Z) = 0$$

$$\text{or,}\ \ g(\mathcal{P}_X Y, Z) - g(A_{JZ}Y, X) = 0.$$

Making use of Proposition 3.3.1, it is seen that

$$g(2A_{JZ}Y - JS(Y, Z), X) = 0,$$

This proves the Theorem completely.

Similarly, for the totally geodesicness of the leaves of $D^\perp$ in M we have:

Proposition 3.3.4 . The leaves of the totally real distribution $D^\perp$ on a CR-submanifold M of an almost Kaehler manifold $\overline{M}$ are totally geodesic in M if and only if

$$g(2A_{JZ}Y - JS(Y,Z), W) = 0$$

for each Y in D and Z, W in $D^\perp$.

Proof. Making use of Proposition 2.2.7 (i) and the Proposition 3.3.1 the proof follows on the same line as that of the above proposition.

From the above two propositions we can say that:

Theorem 3.3.2 . Let M be CR-submanifold of an almost Kaehler manifold $\overline{M}$. Then M is a CR-product if and only if

$$2A_{JD^\perp}D = [JS(D, D^\perp)].$$

Remark. If the manifold $\overline{M}$ is Kaehler, the Nijenthuis tensor is identically zero, therefore in this case M is a CR-product if and only if $A_{JD^\perp}D = 0$, which is precisely the condition obtained by B. Y. Chen [11] as a characterization for CR-products in a Kaehler manifold.

CHAPTER IV

CR-WARPED AND TWISTED PRODUCTS IN KAEHLER MANIFOLDS

4.1 Introduction.

Bishop and O'Neill in 1969 introduced a class of manifolds, namely warped-product manifolds [8]. These are product of Riemannian manifolds with metric defined using a function f on one of the manifold. B.Y.Chen in [16] introduced the notion of CR-warped product submanifold as CR-submanifolds which are warped products of holomorphic and totally real submanifolds of a Kaehler manifold $\overline{M}$. A lot of research is being done on these submanifolds and many results concerning these submanifolds are obtained recently.
Our aim in this chapter is to collect relevent results and also study a more general class viz. twisted product CR-submanifolds of $\overline{M}$.

Recalling from Definition (1.4.7) that if M_1 and M_2 are Riemannian manifolds and $f > 0$ be a smooth function on M_1, the warped product $M = M_1 \times_f M_2$ is defined as the product manifold $M_1 \times M_2$ with metric tensor

$$g = \pi^\star(g_1) + (f \circ \pi)^2 \sigma^\star(g_2),$$

where π and σ are projections of $M_1 \times M_2$ onto M_1 and M_2 respectively and g_1 and g_2 are metrices on M_1 and M_2 respectively. M_1 is called the base of $M = M_1 \times_f M_2$, and M_2 the fiber. We shall express the geometry in terms of warping function and the geometrices of M_1 and M_2. In the case of a semi-Riemannian product it is easy to see that the fibers $p \times M_2 = \pi^{-1}(p)$ and the leaves $M_1 \times q = \sigma^{-1}(q)$ are semi-Riemannian submanifolds of M.

Example(4.1.1): A surface of revolution is a warped product with leaves as the different positions of the rotated curve and fibers the circles of revolution. Explicitly, if M is obtained by revolving a plane curve C about an axis in R^3 and $f : C \longrightarrow R^+$, gives distance to the axis, then the surface $M = C \times_f S^1(1)$ is a warped product manifold.

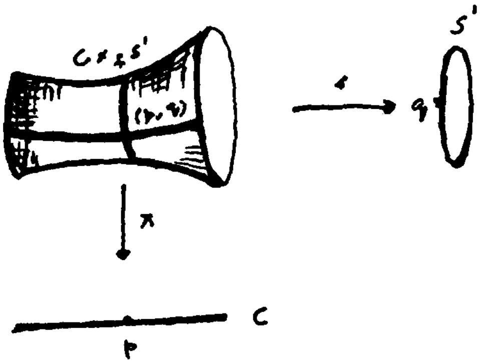

Figure 4.1.1

Example(4.1.2): In spherical coordinates the line element of R^3-$\{0\}$ is

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2 d\phi^2).$$

Setting $r = 1$ gives the line element of the unit sphere S^2. Evidently R^3-$\{0\}$ is diffeomorphic to $R^+ \times S^2$ under the natural map $(t, p) \leftrightarrow tp$. Thus the formula for ds^2 shows that R^3-$\{0\}$ can be identified with the warped product $R^+ \times_r S^2$. In R^3-$\{0\}$ the leaves are the rays from the origin and the fibers are the spheres $S^2(r)$, r>0.

In general, R^3-$\{0\}$ is naturally isometric to $R^+ \times_r S^n$.

4.2 Warped Product CR-submanifolds of Kaehler Manifolds.

The product submanifolds of the form $M = N_\perp \times_f N_T$ or $N_T \times_f N_\perp$ in a Kaehler manifold $\overline{M}$, where $N_\perp$ is totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$ are known as warped product CR-submanifolds. First we shall consider the warped product submanifolds of the form $M = N_\perp \times_f N_T$ of a Kaehler manifold $\overline{M}$. Following theorem [16] implies that in such a case the warped product submanifolds are nothing but CR-product submanifolds. To prove this we need the following:

Lemma 4.2.1 [8]. On a warped product manifold $M = N_\perp \times_f N_T$ we have

$$\nabla_X Z = \nabla_Z X = Z(\ln f)X,$$

for each $X \in T(N_T)$ and $Z \in T(N_\perp)$.

Proof. Taking vector fields X, Y in $T(N_T)$ and Z in $T(N_\perp)$, we find that

$$[X, Z] = 0.$$

Therefore $\nabla_X Z = \nabla_Z X$

Now using (1.2.1) and the fact that N_T and $N_\perp$ are orthogonal to each other, we get

$$\begin{aligned} 2g(\nabla_Z X, Y) &= Zg(X,Y) \\ &= Zf^2 g_{N_T}(X,Y) \\ &= 2f(Zf)g_{N_T}(X,Y). \end{aligned}$$

Because $g(X,Y) = g_{N_\perp}(X,Y) + f^2 g_{N_T}(X,Y)$, (4.2.1)

$$g(\nabla_Z X, Y) = f^2 g_{N_T}(Z(\ln f)X, Y),$$

or $g(\nabla_Z X, Y) = g((Z\ln f)X, Y)$.

Since $g_{N_\perp}(X,Y) = 0$,

$$g(\nabla_Z X - Z(\ln f)X, Y) = 0. \tag{4.2.2}$$

On the other hand, since $M = N_\perp \times_f N_T$ is a warped product, $N_\perp$ is a totally geodesic submanifold of M. Thus, we have

$$g(\nabla_Z X, W) = -g(X, \nabla_z W) = 0. \tag{4.2.3}$$

Combining (4.2.2) and (4.2.3) we get

$$g(\nabla_Z X - Z(\ln f)X, U) = 0, \tag{4.2.4}$$

for a vector field U tangent on M, which implies that

$$\nabla_X Z = \nabla_Z X = Z(\ln f)X.$$

From the above following is easy to observe.

Corollary 4.2.1. Let $M = N_\perp \times_f N_T$ be a warped product manifold then

(i). $N_\perp$ is totally geodesic in M

(ii). N_T is totally umbilical in M.

Theorem 4.2.1 [16]. If $M = N_\perp \times_f N_T$ is a warped product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$, then M is a CR-product.

Proof. Let $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. From the Corollary 4.2.1 we know that in this case $N_\perp$ is a totally geodesic submanifold of M, i.e, $\nabla_Z W \in T(N_\perp)$ for any vector fields Z, $W \in T(N_\perp)$. Using these facts in (1.2.1) we get

$$g([Z,W],Y) = 0. \tag{4.2.5}$$

for any Z, $W \in T(N_\perp)$ and $Y \in T(N_T)$. Moreover if $Y = \xi^i \frac{\partial}{\partial y^i}$ and $Z = \eta^j \frac{\partial}{\partial z^j}$ then

$$[Y,Z] = \xi^i (\frac{\partial}{\partial y^i}\eta^j)\ \frac{\partial}{\partial z^j} - \eta^j (\frac{\partial}{\partial z^j}\xi^i)\ \frac{\partial}{\partial y^i},$$

which shows that $[Y,Z] = 0$ as the ξ^i, and η^j are constant with respect to y^i and z^j respectively.

Similarly

$$[Y,W] = 0. \tag{4.2.6}$$

Using the above two relations in (1.2.1), we get

$$\begin{aligned} 2g(\nabla_Z W, Y) &= -Yg(Z,W) \\ &= -Y[g_{N_\perp}(Z,W) + f^2 g_{N_T}(Z,W)] \\ &= -Y g_{N_T}(Z,W) \\ &= 0, \end{aligned}$$

which implies that $\nabla_Z W \in T(N_\perp)$. Thus for any vector fields Z, W on $N_\perp$ and X on N_T we have

$$g(\nabla_Z W, X) = 0. \tag{4.2.7}$$

Since the ambient space $\overline{M}$ is Kaehler, we have

$$\overline{\nabla}_Z JW = J\overline{\nabla}_Z W.$$

Using Gauss and Weingarten formulas

$$-A_{JW} Z + \nabla^\perp_Z (JW) = J(\nabla_Z W) + Jh(Z,W). \tag{4.2.8}$$

Taking product with JX, we get

$$g(-A_{JW}Z,JX)+g(\nabla_Z^{\perp}(JW),JX)=g(J(\nabla_Z W),JX)+g(Jh(Z,W),JX),$$

$$g(A_{JW}Z,JX)=-g(\nabla_Z W,X). \tag{4.2.9}$$

By combining (4.2.7) and (4.2.9) we obtain

$$g(h(D,D^{\perp}),JD^{\perp})=0. \tag{4.2.10}$$

On the other hand, from Lemma 4.2.1 we have

$$\nabla_X Z=\nabla_Z X=(Z\ln f)X, \tag{4.2.11}$$

for any vector field X in $T(N_T)$ and Z in $T(N_{\perp})$. Thus, if we denote by h^T and A^T the second fundamental form and the shape operator of N_T in M, then by using Gauss and Weingarten formulas we have

$$\begin{aligned} g(h^T(X,Y),Z) &= g(A_Z^TX,Y) \\ &= \text{-}g(\nabla_X Z,Y) \\ &= \text{-}Z(\ln f)g(X,Y), \end{aligned} \tag{4.2.12}$$

for any vectors fields X, Y on N_T and Z on $T(N_{\perp})$. Hence we find that

$$h^T(X,Y)=-grad(\ln f)g(X,Y), \tag{4.2.13}$$

where *grad* denotes the *gradient* operator It follows from equation (4.2.13) implies that N_T is a totally umbilical submanifold of M.

Let $\overline{h}$ denote the second fundamental form of N_T in the ambient space $\overline{M}$. Then

$$\overline{h}(X,Y)=h^T(X,Y)+h(X,Y), \tag{4.2.14}$$

for any X, Y tangent to N_T. By applying (4.2.13) and (4.2.14) we find that

$$g(\overline{h}(X,X),Z)=-Z(\ln f)g(X,X). \tag{4.2.15}$$

Since N_T is a holomorphic submanifold of $\overline{M}$, we also have the following relation

$$\overline{h}(X, JY) = \overline{h}(JX, Y) = J\overline{h}(X, Y). \tag{4.2.16}$$

Hence by combining (4.2.15) and (4.2.16), we obtain

$$g(\overline{h}(X, X), Z) = -g(\overline{h}(JX, JX), Z) = Z(\ln f)g(X, X). \tag{4.2.17}$$

(4.2.15) and (4.2.17) imply that $Z(\ln f) = 0$. Therefore by (4.2.12) and (4.2.14)

$$g(\overline{h}(X, Y), Z) = g(h^T(X, Y), Z), \tag{4.2.18}$$

for any X, Y in D and Z in $D^{\perp}$. Hence by (4.2.14) (4.2.16) and (4.2.18)

$$g(h(X, Y), JZ) = g(h(X, Y), JZ) = -g(h(X, JY), Z) = 0.$$

Therefore

$$g(h(D, D), JD^{\perp}) = 0. \tag{4.2.19}$$

(4.2.10) and (4.2.19) imply that

$$A_{JD^{\perp}}D = 0.$$

Therefore, by applying (2.3.7), we conclude that $M = N_{\perp} \times_f N_T$ is CR-product.

As from above we can see that proper warped product CR-submanifold of a Kaehler manifold are trivial, we shall now consider warped products of the type $N_T \times_f N_{\perp}$ and also discuss a simple characterization for these kinds of CR-warped product submanifolds.

Lemma 4.2.2 [16]. For a CR-warped product $M = N_T \times_f N_\perp$ in any Kaehler manifold $\overline{M}$, we have

(i). $g(h(D, D), JD^\perp) = 0$;

(ii). $\nabla_X Z = \nabla_Z X = X(\ln f)Z$;

(iii). $g(h(JX, Z), JW) = X(\ln f)g(Z, W)$;

(iv). $\nabla_X^\perp(JZ) = J\nabla_X Z$, whenever $h(D, D^\perp) \subset JD^\perp$;

(v). $g(h(D, D^\perp), JD^\perp) = 0$; if and only if $M = N_T \times_f N_\perp$ is a trivial CR-warped product in $\overline{M}$

where X, Y are vector fields on N_T and Z, W on $N_\perp$.

Proof. Since $\overline{M}$ is Kaehler, we have

$$\overline{\nabla}_X JZ = J\overline{\nabla}_X Z,$$

$$A_{JZ}X + \nabla_X^\perp JZ = J\nabla_X Z + Jh(X, Z), \tag{4.2.20}$$

for any vectors fields X, Y on N_T and Z in $N_\perp$. Thus, by taking the inner product of (4.2.20) with JY, we find

$$g(-A_{JZ}X, JY) + g(\nabla_X^\perp JZ, JY) = g(J\nabla_X Z, JY) + g(Jh(X, Z), JY),$$

i.e., $g(-A_{JZ}X, JY) = g(\nabla_X Z, Y)$,

or $-g(h(X, JY), JZ) = g(\nabla_X Z, Y)$.

(4.2.21)

On the other hand, Since $M = N_T \times_f N_\perp$ is a warped product, N_T is a totally geodesic submanifold of M. Thus, we also have $g(\nabla_X Z, Y) = 0$. Combining this with (4.2.21), we get

$$g(h(D, D), JD^\perp) = 0.$$

This prove statement (1).

Statement (2) is already prove in Lemma 4.2.1

Now if we denote by h and A the second fundamental form and the shape operator of the immersion of M in $\overline{M}$. Then we obtain from the formula of Gauss and Weingarten that

$$g(h(JX,Z),JW) = -g(JA_{JW}Z,X). \tag{4.2.22}$$

On using Lemma 4.2.1 and statement (2) we get

$$\begin{aligned} g(h(JX,Z),W) &= -g(\nabla_Z W,X) \\ &= g(\nabla_Z X,W) \\ &= X(\ln f)g(Z,W). \end{aligned}$$

for any X in $T(N_T)$ and Z,W in $T(N_\perp)$. This proves statement (3).

Since N_T is a totally geodesic submanifold in M, $\nabla_X Z \in D^\perp$. Thus $J\nabla_X Z \in JD^\perp$. On the other hand, condition $h(D,D^\perp) \subset JD^\perp$ implies $Jh(X,Z) \in TM$. Therefore by aplying (4.2.20), we obtain statement (4).

If $g(h(D,D^\perp),JD^\perp) = 0$, then by statement (3)

$$(X \ln f) = 0.$$

That means $M = N_T \times_f N_\perp$ is CR-product.

Conversely, if M is a CR-product submanifold then by Lemma 2.3.6

$$g(h(D,D^\perp),JD^\perp) = 0.$$

This implies statement (5).

Now we will give the following simple characterization of CR-warped products.

Theorem 4.2.2 [16]. A proper CR-submanifold M of a Kaehler manifold $\overline{M}$ is locally a CR-warped product if and only if

$$A_{JZ}X = ((JX)\mu)Z \tag{4.2.23}$$

for $X \in D$ and $Z \in D^{\perp}$ and for some function μ on M satisfying $W\mu = 0$ where $W \in D^{\perp}$.

Proof. If M is a CR-warped product $N_T \times_f N_{\perp}$ in a Kaehler manifold $\overline{M}$, then statement (1) and (2) of Lemma 4.2.2 imply that $A_{JZ}X = -((JX)\ln f)Z$ for each $X \in D$ and $Z \in D^{\perp}$. Since f is a function on N_T, we also have $W(\ln f) = 0$ for all $W \in D^{\perp}$.

Conversely, assume that M is a proper CR-submanifold of a Kaehler manifold $\overline{M}$ satisfying

$$A_{JZ}X = ((JX)\mu)Z \qquad (4.2.24),$$

for $X \in D$, $Z \in D^{\perp}$ and for some function μ with $W\mu = 0$, $W \in D^{\perp}$ thus we have

$$g(h(D,D), JD^{\perp}) = 0 \tag{4.2.25}$$

It follows from (4.2.25) that the holomorphic distribution D is intergrable and its leaves are totally geodesic in M. Also

$$\begin{aligned} g(((J^2X)\mu)Z, W) &= g(((-X)\mu)Z, W) \\ &= g(A_{JZ}JX, W) \\ &= g(-\overline{\nabla}_{JX}JZ, W \\ &= g(J\overline{\nabla}_{JX}Z, W). \end{aligned}$$

Therefore

$$-X(\mu)g(Z,W) = g(\overline{\nabla}_{JX}Z, JW)$$

$$= g(h(JX, Z), JW). \quad (4.2.26)$$

On the other hand, from Lemma 4.2.1 and (4.2.26) we have

$$\begin{aligned} g(\nabla_Z X, W) &= -g(\nabla_Z W, X) \\ &= -g(JA_{JW}Z, X) \\ &= g(h(JX, Z), JW) \\ &= -X(\mu)g(Z, W). \quad (4.2.27) \end{aligned}$$

If we denote by $\nabla_Z^{\perp}$ the connection and $h^{\perp}$ as the second fundamental form of $N^{\perp}$ in M then

$$g(X, \nabla_Z W) = g(X, \nabla_Z{}^{\perp} W) + g(X, h^{\perp}(Z, W))$$

or $$g(X, \nabla_Z W) = g(X, h^{\perp}(Z, W)).$$

From (4.2.27)

$$\begin{aligned} g(\text{X}, \text{h}^{\perp}(\text{Z,W})) &= \text{-X}(\mu)\, g(\text{ Z,W}) \\ &= \text{-}g(grad(\mu), X)\; g(Z, W) \\ &= -g(X, g(Z, W)grad(\mu)). \quad (4.2.28) \end{aligned}$$

On the other hand, since $h^{\perp}(Z, W)$ is a vector normal to $D^{\perp}$ and tangential to M,

$$g(Z', h^{\perp}(Z, W)) = 0, \quad (4.2.29)$$

for any vector field Z' on $D^{\perp}$. Combining (4.2.28) and (4.2.29) we get

$$g(U, h^{\perp}(Z, W) - g(Z, W)grad(\mu)) = 0$$

$$h^{\perp}(Z, W) = -g(Z, W)grad(\mu), \quad (4.2.30)$$

for $X \in D$ and Z, $W \in D^{\perp}$. Using Weingarten formula we have

$$\nabla_W H = -A_H W + \nabla_W{}^{\perp} H,$$

for vector field W tangent on $N^{\perp}$ and H normal on $N^{\perp}$

$$\begin{aligned} g(\nabla_H W, X) &= g(\nabla_W H, X) \\ &= g(-A_H W, X) + g(\nabla_W{}^{\perp} H, X) \end{aligned}$$

$$-g(W, \nabla_H X) = g(-A_H W, X) + g(\nabla_W{}^{\perp} H, X). \qquad (4.2.31)$$

Since D is totally geodesic in M,

$$g(W, \nabla_H X) = 0.$$

Hence

$$g(\nabla_W{}^{\perp} H, X) = 0. \qquad (4.2.32)$$

Also

$$g(\nabla_W{}^{\perp} H, Z) = 0. \qquad (4.2.33)$$

Since Z is a vector field tangent on $N_{\perp}$ and $\nabla^{\perp}_{W'} H$ normal on $N_{\perp}$combining (4.2.32) and (4.2.33), we obtain

$$g(\nabla_W{}^{\perp} H, U) = 0$$

$$\nabla_{W'}{}^{\perp} H = 0 \qquad (4.2.34).$$

Since the totally real distribution $D^{\perp}$ of a CR-submanifold of a Kaehler manifold is always integrable (4.2.30) and (4.2.24) imply that each leaf of $D^{\perp}$ is an extrinsic sphere in M, i.e, a totally umbilical submanifold with parallel mean curvature vector. Thus M is locally warped product$N_T \times_f N_{\perp}$ of a holomorphic submanifold and totally real submanifold N_T of M, where N_T is a leaf of D and $N_{\perp}$ is a leaf of $D_{\perp}$ and f is a certain warping function (cf.[11]).

4.3 A General Inequality for CR-warped Product Submanifolds.

In this section we shall give a general inequality for CR-warped product submanifolds in a Kaehler manifold proved by Chen [16].

Definition (4.3.1). For a real hyperspace M of a Kaehler manifod $\overline{M}$ with a unit normal vector fild ξ, the tangent vector field $J\xi$ on M is called a characteristic vector field of M.

Definition (4.3.2). A unit tangent vector V on M is called a principal vector if V is an eigenvector of the shape operator A_ξ, the corresponding eigenvalue is called the principal curvature at V.

We have the following result for CR-warped products in Kaehler manifolds.

Theorem 4.3.1 [16]. Let $M = N_T \times_f N_\perp$ be a CR-warped product submanifold in a Kaehler manifold $\overline{M}$. We have

(i) The squared norm of the second fundamental form of M satisfies

$$||h||^2 \geq 2p||grad(\ln f)||^2, \tag{4.3.1}$$

where p is the dimension of $N_\perp$.

(ii) If the equality sign of (4.3.1) holds identically, then N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\overline{M}$. Moreover, M is a minimal submanifold in $\overline{M}$.

(iii) When M is anti-holomorphic and $p > 1$, the equality sign of (4.3.1) holds identically if and only if $N_\perp$ is a totally umbilical submanifold of $\overline{M}$.

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(iv) If M is anti-holomorphic and p = 1, then the equality sign of (4.3.1) holds identically if and only if the characteristic vector field $J\xi$ of M is a principal vector field with zero as its principal curvature.(Note that in this case, M is a real hypersurface in $\overline{M}$.) Also, in this case, the equality sign in (4.3.1) holds identically if and only if M is a minimal hypersurface in $\overline{M}$.

Proof. From Lemma 4.2.1, we have

$$g(h(Z,JX),JZ) = X(\ln f), \tag{4.3.2}$$

where Z is any unit vector in $D^{\perp}$. Applying (4.3.2) we obtain inequality (4.3.1) immediately.

For any vector fields X in D and Z, W in $D^{\perp}$, Lemma 2.3.3 and (1.3.4) imply that

$$g(\nabla_W Z, X) = g(JA_{JZ}W, X) = -g(h(JX,W),JZ). \tag{4.3.3}$$

Hence by using Lemma 4.2.1 and (4.3.3), we find

$$g(\nabla_W Z, X) = -(X \ln f)g(Z,W). \tag{4.3.4}$$

On the other hand, if we denote by $h^{\perp}$ the second fundamental form of $N_{\perp}$ in

$M = N_T \times_f N_{\perp}$ we get

$$g(h^{\perp}(Z,W),X) = g(\nabla_W Z, X) \quad . \tag{4.3.5}$$

Combining (4.3.4) and (4.3.5) we obtain,

$$h^{\perp}(Z,W) = -g(Z,W)grad(\ln f). \tag{4.3.6}$$

Now, assume that the equality case of (4.2.1) holds, then we obtain from (4.2.2) that

$$h(D,D) = 0, \qquad h(D^\perp, D^\perp), \qquad h(D, D^\perp) \subset JD^\perp. \tag{4.3.7}$$

Since N_T is a totally geodesic submanifold in M, the first condition in (4.3.7) implies that N_T is totally geodesic in $\overline{M}$.

On the other hand, (4.3.6) shows that $N_\perp$ is totally umbilical in M. Now the second condition in (4.3.7) implies that $N_\perp$ is also totally umbilical in $\overline{M}$. Moreover, from (4.3.7), we know that M is minimal in $\overline{M}$.

Let us assume that M is an anti-holomorphic CR-wraped product in $\overline{M}$. Then, from statement (i) of Lemma 4.2.2, we get

$$h(D,D) = 0. \tag{4.3.8}$$

If $N_\perp$ is totally umbilical in $\overline{M}$, then there exists a normal vector field $\overline{H}$ of $N_\perp$ in $\overline{M}$ such that the second fundamental form $\overline{h}$ of $N_\perp$ in $\overline{M}$ satisfies

$$\overline{h}(Z,W) = g(Z,W)\overline{H}, \tag{4.3.9}$$

for Z, W tangent to $N_\perp$. Since

$$\overline{h}(Z,W) = h^\perp(Z,W) + h(Z,W),$$

(4.3.9) implies that there is a normal vector field η such that

$$h(Z,W) = g(Z,W)\eta, \tag{4.3.10}$$

for each unit vector $W \in D^T$ and each unit vector Z in $D^\perp$ perpendicular to W, we have

$$
\begin{aligned}
g(\eta, JW) &= g(h(W,Z), JW) \\
&= g(h(Z,W), JZ) \\
&= g(Z,W)\, g(\eta, JZ) \\
&= 0. \qquad (4.3.11)
\end{aligned}
$$

Since M is assumed to be anti-holomorphic, (4.3.11) implies either $p = 1$ or

$$h(D^{\perp}, D^{\perp}) = 0. \qquad (4.3.12)$$

Hence, (4.3.2), (4.3.8) and (4.3.12) implies the equality case of (4.3.1) holds whenever $p>1$.

When $p = 1$, M is a real hypersurface of $\overline{M}$. In this case, the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature if and only (4.3.12) holds. So, in this case we also have equality case of (4.3.1) if the characteristic vector field $J\xi$ is a principal vector field with zero as its principal curvature. From the first condition in (4.3.7), we also know that condition (4.3.12) holds if and only if M is minimal in $\overline{H}$ By applying statement (ii), the converse is easy to verify.

4.4 Twisted Product CR-submanifolds.

Twisted product manifolds are a natural generalization of warped product manifolds as is evident from the following definition

Definition (4.4.1). Suppose M_1 and M_2 are Riemannian manifolds and let $f > 0$ be a smooth function on $M_1 \times M_2$. The warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ with metric tensor

$$g = \pi^{\star}(g_1) + (f \circ \pi)^2 \sigma^{\star}(g_2),$$

where π and σ are projections of $M_1 \times M_2$ onto M_1 and M_2 respectively and

g_1 and g_2 are Riemannian metrices on M_1 and M_2 respectively.
It can be seen when the twisted function f depends only on M_1, the twisted product $M = M_1 \times_f M_2$ reduces to a warped product.

As was the case for warped products we shall consider the twisted products of the form $M = N_\perp \times_f N_T$ immeresed in a Kaehler manifold $\overline{M}$, where $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$ and show that such type of twisted products are nothing but CR-products.

Theorem 4.4.1 [18]. If $M = N_\perp \times_f N_T$ is a twisted product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$, then M is a CR-product.

Proof. Assume that if $M = N_\perp \times_f N_T$ is a twisted product CR-submanifold in a Kaehler manifold $\overline{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. The metric tensor g of M is then given by

$$g = g_{N_\perp} + f^2 g_{N_T}. \tag{4.4.1}$$

From (4.4.1) it follows that $N_\perp$ is a totally geodesic submanifold of M. Thus, for any vector fields Z, W on $N_\perp$ and X on N_T, we have

$$g(\nabla_Z W, X) = 0. \tag{4.4.2}$$

Since the ambient space $\overline{M}$ is Kaehler, $\overline{\nabla}_Z(JW) = J\overline{\nabla}_Z W$. Thus, from the Gauss and Weingarten formulas, we obtain

$$-A_{JW}Z + \nabla^\perp_Z(JW) = J(\nabla_Z W) + Jh(Z, W). \tag{4.4.3}$$

Taking the inner product of (4.4.3) with JX, we have

$$g(A_{JW}Z, JX) = -g(J\nabla_Z W, JX).$$

or

$$g(A_{JW}Z, JX) = -g(\nabla_Z W, X). \tag{4.4.4}$$

By combining (4.4.2) and (4.4.4) we obtain

$$g(h(D, D^{\perp}, JD^{\perp}) = 0, \tag{4.4.5}$$

where D and $D^{\perp}$ are the holomorphic distribution and the totally real distribution of M, respectively.

On the other hand, since $[X, Z] = 0$ for any vector field X in D and Z in $D^{\perp}$, we have

$$\nabla_X Z = \nabla_Z X. \tag{4.4.6}$$

Thus, for any vector fields X, Y in D and Z in $D^{\perp}$, we get

$$\begin{aligned} Zg(X,Y) &= Zg_{N_T}(f^2(X,Y)) \\ &= 2f(Zf)g_{N_T}(X,Y) \\ &= 2(Zf/f)g(X,Y), \end{aligned} \tag{4.4.7}$$

Also, from (4.4.6) and $g(X, Z) = 0$ we find

$$\begin{aligned} Zg(X,Y) &= g(\nabla_Z X,Y)+g(X,\nabla_Z Y) \\ &= g(\nabla_X Z,Y)+g(X,\nabla_Y Z) \\ &= g(\nabla_X Z,Y)-g(\nabla_Y X,Z). \end{aligned} \tag{4.4.8}$$

If we denote by h^T and A^T the second fundamental form and the shape operator of N_T in M, then from the Gauss and Weingarten formulae and from (4.4.8) we obtain

$$Zg(X,Y) = -2g(h^T(X,Y), Z). \tag{4.4.9}$$

Combining (4.4.7) and (4.4.9) we get

$$h^T(X,Y) = -grad^{\perp}(\ln f)g(X,Y), \tag{4.4.10}$$

where $grad^{\perp}(\ln f)$ denotes the N_T-component of the gradient $grad(\ln f)$ of $\ln f$.

Equation (4.4.10) implies that N_T is totally unbilical in M.

Let $\overline{h}$ denote the second fundamental form of N_T in the ambient space $\overline{M}$. Then we have

$$\overline{h}(X,Y) = h^T(X,Y) + h(X,Y), \tag{4.4.11}$$

for any X,Y tangent to N_T. By applying (4.4.10) and (4.4.11) we find

$$\begin{aligned} g(\overline{h}(X,X),Z) &= g(h^T(X,X),Z) \\ &= -Z(\ln f)g(X,X). \end{aligned} \tag{4.4.12}$$

On the other hand, since N_T is a holomorphic submanifold of $\overline{M}$, we also have the following relations:

$$\overline{h}(X,JY) = \overline{h}(JX,Y) = J\overline{h}(X,Y). \tag{4.4.13}$$

Hence, by combining (4.4.12) and (4.4.13),we obtain

$$\begin{aligned} g(\overline{h}(X,X),Z) &= g(\overline{h}(JX,JX),Z) \\ &= -Z(\ln f)g(X,X). \end{aligned} \tag{4.4.14}$$

Therefore, we obtain $Z(\ln f) = 0$ for any Z tangent to $N_{\perp}$. Hence, we find from (4.4.10) that

$$g(\overline{h}(X,Y),Z) = g(h^T(X,Y),Z) = 0 \tag{4.4.15}$$

for any X, Y in D and Z in $D^{\perp}$.

On the other hand, since N_T is a holomorphic submanifold of $\overline{M}$, we also have

$$\overline{h}(X,JY) = \overline{h}(JX,Y) = J\overline{h}(X,Y). \tag{4.4.16}$$

Hence, by (4.4.11), (4.4.15) and (4.4.16), we obtain

$$\begin{aligned} g(h(X,Y),JZ) &= g(\overline{h}(X,Y),JZ) \\ &= -g(\overline{h}(X,JY),Z) \\ &= 0. \end{aligned}$$

Therefore

$$g(h(D,D),JD^{\perp}) = 0. \qquad (4.4.17)$$

Equation (4.4.5) and (4.4.17) imply that $A_{JD^{\perp}}D = 0$. Therefore, from 2.3.7 to conclude that the CR-submanifolds $M = N_T \times_f N_{\perp}$ is a CR-product.

4.5 Twisted Products $N_T \times_f N_{\perp}$ of Kaehler Manifolds.

In this section we study the class of CR-submanifold in Kaehler manifold which are twisted product of the form $N_T \times_f N_{\perp}$, where N_T is a totally real submanifold and $N_{\perp}$ is a holomorphic submanifold of $\overline{M}$. First we shall prove an inequality similar to the one proved in the previous article.

Theorem 4.5.1. Let M = $N_T \times_f N_{\perp}$ be a twisted product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_{\perp}$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. Then we have

(i) The squared norm of the second fundamental form of M in $\overline{M}$ satisfies

$$||h||^2 \geq 2p||grad^T(\ln f)||^2 \qquad (4.5.1)$$

Where $grad^T(\ln f)$ is the N^T-component of the gradiant of $\ln f$ and p is the dimension of $N_{\perp}$.

(ii) If $||h||^2 = 2p||grad^T(\ln f)||^2$ holds identically, then N_T is a totally geodesic submanifold and $N_{\perp}$ is a totally umbilical submanifold of $\overline{M}$.

(iii) If M is anti-holomorphic and $\dim N_{-} > 1$, then $||h||^2 = 2p||grad^T(\log f)||^2$ holds identically if and only if N_T is a totally geodesic submanifold and $N_\perp$ is a totally umbilical submanifold of $\overline{M}$.

Proof. Since $\overline{M}$ is Kaehler, $\overline{\nabla}J = 0$. Thus, for any vector fields X,Y in D and Z in $D^\perp$, we have

$$J\nabla_X Z + Jh(X,Z) = -A_{JZ}X + \nabla_X^\perp JZ.$$

Therefore, by taking the inner product of this equation with JY, we find

$$\begin{aligned} g(\nabla_X Z, Y) &= -g(A_{JZ}X, JY) \\ &= -g(h(X, JY), JZ). \end{aligned} \tag{4.5.2}$$

Since $M = N_T \times_f N_\perp$ is a twisted product, N_T is a totally geodesic submanifold of M. Therefore, we also have $g(\nabla_X Z, Y) = 0$. Combining this equation with (4.5.2), we obtain

$$g(h(D,D), JD^\perp) = 0. \tag{4.5.3}$$

We know that the second fundamental form $h^\perp$ of $N_\perp$ in M is given by

$$h^\perp(Z,W) = -grad^T(\ln f)g(Z,W), \tag{4.5.4}$$

for vector fields Z, W in $D^\perp$. From (4.5.4) we get

$$g(\nabla_Z W, X) = -X(\ln f)g(Z,W). \tag{4.5.5}$$

On the other hand, from Lemma 2.3.3, we also have

$$g(JA_{JW}Z, X) = g(\nabla_Z W, X). \tag{4.5.6}$$

Hence, by combining (4.5.5) and (4.5.6) we have

$$\begin{aligned} g(h(JX,Z),JW) &= -g(JA_{JW}Z,X) \\ &= X(\ln f)g(Z,W), \end{aligned} \tag{4.5.7}$$

for X in D and Z, W in $D^{\perp}$.

By (4.5.7) and Lemma 2.3.3 we know that, for any unit vector field Z in $D^{\perp}$, we have

$$g(h(Z,JX),JZ) = g(h(JX,Z),JZ) = X\ln f. \tag{4.5.8}$$

Applying (4.5.8) we obtain inequality (4.5.1).

For vector fields X in D and Z,W in $D^{\perp}$, we obtain from Lemma 2.3.3 that

$$g(\nabla_W Z,X) = g(JA_{JZ}W,X) = -g(h(JX,W),JZ). \tag{4.5.9}$$

Hence, by using (4.5.7) and (4.5.9), we obtain

$$g(\nabla_W Z,X) = -(X\ln f)g(Z,W). \tag{4.5.10}$$

On the other hand, if we denote by $h^{\perp}$ the second fundamental form of $N_{\perp}$ in $M = N_T \times_f N_{\perp}$, we get

$$g(h^{\perp}(Z,W),X) = g(\nabla_W Z,X). \tag{4.5.11}$$

Combining (4.5.10) and (4.5.11) yields

$$h^{\perp}(Z,W) = -g(Z,W)\nabla^T(lnf). \tag{4.5.12}$$

Now, suppose that the equatility case of (4.5.1) holds identically. Then we obtain from (4.5.7) that

$$h(D,D) = 0, \quad h(D^{\perp},D^{\perp}) = 0, \quad h(D,D^{\perp}) \subset JD^{\perp}. \tag{4.5.13}$$

Since N_T is totally geodesic submanifold of the twisted product manifold $M = N_T \times_f N_\perp$, the first condition in (4.5.13) implies that N_T is a totally geodesic submanifold in the ambient space $\overline{M}$. Equation (4.5.12) implies that $N_\perp$ is a totally umbilical submanifold in M and from the second condition in (4.5.13) we conclude that $N_\perp$ is also a totally umbilical submanifold in $\overline{M}$.

In order to prove the last statement, let us assume that M is anti-holomorphic submanifold of $\overline{M}$ and dim $N_\perp > 1$. If $N_\perp$ is totally umbilical submanifold of $\overline{M}$, then there exist a normal vector field $\overline{H}$ of $N_\perp$ in $\overline{M}$ such that the second fundamental form $\overline{h}$ of $N_\perp$ in $\overline{M}$ is given by

$$\overline{h}(Z, W) = g(Z, W)\overline{H}, \tag{4.5.14}$$

for all vectors Z, W tangent to

$$\overline{h}(Z, W) = \overline{h}(X, W) + h(Z, W), \tag{4.5.15}$$

(4.5.13), (4.5.14) and (4.5.15) imply

$$h(Z, W) = g(Z, W)\eta, \tag{4.5.16}$$

for some normal vector field η of M in $\overline{M}$. Therefore, for any given unit vector W in $D^\perp$ and any given unit vector Z in $D^\perp$ perpendicular to W, we have

$$g(\eta, JW) = g(h(Z, Z), JW) = g(h(Z, W), JZ) = g(Z, W)g(\eta, JZ) = 0. \tag{4.5.17}$$

We have applied Lemma 2.3.3. Hence, we obtain

$$h(Z, W) \in \mu. \tag{4.5.18}$$

On the other hand, since M is assumed to be an anti-holomorphic submanifold of $\overline{M}$, we have $\mu = 0$. Hence

$$h(D^\perp, D^\perp) = 0. \tag{4.5.19}$$

Therefore, by using (4.5.7) and from the assumption $TM^{\perp} = JD^{\perp}$, we get

$$h(JX, W) = (X \ln f)JW. \tag{4.5.20}$$

Finally, if we assume that N_T is a totally geodesic submanifold of $\overline{M}$, we also have

$$h(D, D) = 0. \tag{4.5.21}$$

By applying (4.5.19), (4.5.20) and (4.5.21), we obtain the equality $||h||^2 = 2p||grad^T(\ln f)||^2$ identically.

The converse of this was already proved while proving statement (ii) of the theorem.

Theorem 4.5.2 [18]. Let $M = N_T \times_f N_{\perp}$ be a twisted product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_{\perp}$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. If M is mixed totally geodesic, then we have

(i) The twisted function f is a function on $N_{\perp}$.

(ii) $N_T \times N_{\perp}^f$ is a CR-product submanifold, where $N_{\perp}^f$ denotes the manifold $N_{\perp}$ equipped with the metric $g^f_{N_{\perp}} = f^2 g_{N_{\perp}}$.

Proof. Let $M = N_T \times_f N_{\perp}$ be a twisted product CR-submanifold of a Kaehler manifold $\overline{M}$ such that $N_{\perp}$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. Then

$$g(h(JX, Z), JW) = -g(JA_{JW}Z, X) = X(\ln f)g(Z, W). \tag{4.5.22}$$

where X in D and Z, W in $D^{\perp}$. Therefore, if M is mixed totally geodesic, we have $X(\ln f) = 0$ for any vector X tangent to N^T. Hence the twisted function of the twisted product depends only on the second factor $N_\perp$. Clearly, in this case the twisted product $N_T \times_f N_\perp$ is isometric to the Riemannian product $N_T \times N_\perp^f$. Hence, with respect to the metric $g^f_{N_\perp} = f^2 g_{N_\perp}$ on $N_\perp$, $N_T \times N_\perp^f$ becomes a CR-product submanifold of $\overline{M}$.

CHAPTER V

RECENT PROGRESS ON WARPED PRODUCT SUBMANIFOLDS

5.1 Introduction.

In the previous chapter we have defined and given characterizations and properties of warped product submanifolds and seen characterizations for them to become CR-product submanifolds. In this last chapter of our dissertation we shall discuss recent progress on warped product submanifolds. We shall first give results by B. sahin [36] on the non-existence of warped product semi-slant submanifolds of a Kaehler manifold and then prove the extensions of these results as well as those of B.Y. Chen by K.A Khan, V.A. Khan *et. al.*[28] in the nearly Kaehler settings.

5.2 Warped Product Semi-slant Submanifolds of Kaehler Manifolds.

First recalling the definition of semi-slant submanifolds of a manifolds M of an almost Hermition manifold $\bar{M}$ as follows:

Definition (5.2.1). A submanifold M of an almost Hermition manifold $\overline{M}$is called semi-slant if it is endowed with two orthogonal distributions D and $D^{\perp}$, where D is invariant with respect to J and $D^{\perp}$ is slant. From (1.4.7) it is easy to see that, M is a slant submanifold of $\overline{M}$ if and only if

$$P^2 = \lambda I, \qquad (5.2.1)$$

for some real number $\lambda \in [-1, 0]$.

Lemma 5.2.1 . Let $M = M_\theta \times_f M_T$ be a warped product submanifold in a Kaehler manifold $\overline{M}$ then we have

$$g(FZ, h(JX, JY)) = PZ(\ln f)g(X, Y). \tag{5.2.2}$$

Proof. For $X \in T(M_T)$ and $Z \in T(M_\theta)$.

$$g(\nabla_{JX}X, Z) = 0.$$

Using (1.2.2) and Gauss formula we get

$$g(JZ, \overline{\nabla}_{JX}JX) = 0.$$

Then from (1.4.7) we have

$$g(PZ + FZ, \overline{\nabla}_{JX}JX) = 0.$$

Now Gauss formula implies

$$-g(\nabla_{JX}PZ, JX) + g(FZ, h(JX, JX)) = 0.$$

Using Lemma 4.2.1 we obtain

$$g(FZ, h(JX, JX)) = PZ(\ln f)g(X, X).$$

Thus we get

$$g(FZ, h(JX, JY)) = PZ(\ln f)g(X, Y),$$

for $X, Y \in T(M_T)$ and $Z \in T(M_\theta)$, which proves the Lemma completely.

Lemma 5.2.2 . Let $M = M_\theta \times_f M_T$ be a warped product submanifold in a Kaehler manifold $\overline{M}$ then we have

$$g(h(JX, JY), FZ) = Z(\ln f)g(JX, Y) + PZ(\ln f)g(X, Y). \tag{5.2.3}$$

Proof. Using Weingarten formula we have

$$g(A_{FZ}JX, JY) = -g(\overline{\nabla}_{JX}FZ, JY),$$

for $X, Y \in T(M_T)$ and $Z \in T(M_\theta)$

$$g(A_{FZ}JX, JY) = g(FZ, \overline{\nabla}_{JX}JY).$$

Thus, using (1.4.7) and (1.2.2) we get

$$\begin{aligned} g(A_{FZ}JX, JY) &= g(JZ - PZ, \overline{\nabla}_{JX}JY) \\ &= g(Z, \overline{\nabla}_{JX}Y) - g(PZ, \overline{\nabla}_{JX}JY). \end{aligned}$$

Thus from Lemma 4.2.1 we have

$$\begin{aligned} g(A_{FZ}JX, JY) &= g(Z, \overline{\nabla}_{JX}Y) + g(\overline{\nabla}_{JX}PZ, JY \\ &= g(\nabla_{JX}Z, Y) + g(\nabla_{JX}PZ, JY) \\ &= Z(\ln f)g(JX, Y) + PZ(\ln f)g(X, Y). \end{aligned}$$

Therefore, using (1.3.4) we obtain

$$g(h(JX, JY), FZ) = Z(\ln f)g(JX, Y) + PZ(\ln f)g(X, Y).$$

This proves the Lemma completely.

Theorem 5.2.1 [36]. Let $\overline{M}$ be a Kaehler manifold. Then there do not exist warped-product submanifolds $M = M_\theta \times_f M_T$ in $\overline{M}$ such that M_θ is a proper slant submanifold and M_T is a holomorphic submanifold of $\overline{M}$.

Proof. By the definition of semi-slant submanifolds and using Lemma 4.2.1 we have

$$g(\nabla_{JX}Z, X) = Z(\ln f)g(JX, X) = 0.$$

Lemma 5.2.1 gives

$$g(FZ, h(JX, JY)) = PZ(\ln f)g(X, Y), \tag{5.2.4}$$

for $X, Y \in T(M_T)$ and $Z \in T(M_\theta)$. Now from Lemma 5.2.2 we have

$$g(h(JX, JY), FZ) = Z(\ln f)g(JX, Y) + PZ(\ln f)g(X, Y), \tag{5.2.5}$$

for $X, Y \in T(M_T)$ and $Z \in T(M_\theta)$. Thus (5.2.4) and (5.2.5) impliy that

$$Z(\ln f)g(JX, Y) = 0,$$

for X, $Y \in T(M_T)$ and $Z \in T(M_\theta)$. Thus

$$Z(\ln f) = 0,$$

which shows that f is constant. Thus proof is complete.

Now, we take up the warped product semi-slant submanifolds in the form of $M_T \times_f M_\theta$ such that M_T is a holomorphic submanifold and M_θ is a proper slant submanifold of $\overline{M}$, and show that there does not exist semi-slant warped products of the type $M = M_T \times_f M_\theta$ in Kaehler manifolds either.

Lemma 5.2.3 . Let $M = M_\theta \times_f M_T$ be a warped product submanifold in a Kaehler manifold $\overline{M}$ then

$$\begin{aligned} &\text{(i)} \quad & X(\ln f)cos^2\theta g(Z, Z) &= -g(h(PZ, X), FZ), \quad (\\ &\text{(ii)} \quad & g(h(PZ, X), FZ) &= g(h(Z, X), FPZ) = X(\ln f)cos^2\theta g(Z, Z), (\end{aligned}$$

for $X, Y \in T(M_T)$ and $Z \in T(M_\theta)$.

Proof. For $X \in T(M_T)$ and $Z \in T(M_\theta)$. Since $T(M_T)$ and $T(M_\theta)$ are orthogonal, we obtain

$$g(\overline{\nabla}_{PZ}X, Z) = -g(\overline{\nabla}_{PZ}Z, X) = 0.$$

Using (1.2.2) we get

$$g(JX, \overline{\nabla}_{PZ}JZ) = 0.$$

Then from (1.4.7), (1.3.4) and Weingarten formula we obtain

$$\begin{aligned} g(JX, \overline{\nabla}_{PZ}PZ + FZ) &= g(JX, \overline{\nabla}_{PZ}TZ) + g(JX, \overline{\nabla}_{PZ}FZ) \\ &= -g(\nabla_{PZ}JX, PZ) - g(JX, A_{FZ}PZ) \\ &= g(\nabla_{PZ}JX, PZ) + g(h(PZ, JX), FZ) \\ &= 0. \end{aligned}$$

Thus using Lemma 4.2.1 and (1.4.13) we get

$$JX(\ln f)cos^2\theta g(Z, Z) = g(h(PZ, JX), FZ), \tag{5.2.8}$$

for X in $T(M_T)$ and Z in $T(M_\theta)$. Subtituting X by JX in (5.2.8) we arrive at

$$X(\ln f)cos^2\theta g(Z, Z) = -g(h(PZ, X), FZ). \tag{5.2.9}$$

Also subtituting Z by PZ in (5.2.9) and using (5.2.1) and (1.4.13) we obtain

$$g(h(P^2Z, X), FPZ) = -X(\ln f)cos^2\theta g(PZ, PZ),$$

$$cos^2\theta g(h(Z, X), FPZ) = X(\ln f)cos^4\theta g(Z, Z).$$

Hence we have

$$g(h(Z, X), FPZ) = X(\ln f)cos^2\theta g(Z, Z), \tag{5.2.10}$$

for X in $T(M_T)$ and Z in $T(M_\theta)$.

Now from Gauss formula we have

$$g(h(PZ, X), FW) = g(\overline{\nabla}_X PZ, FW),$$

for X in $T(M_T)$ and Z, W in $T(M_\theta)$. Hence,

$$g(h(PZ, X), FW) = g(PZ\overline{\nabla}_X FW).$$

Thus using (1.4.7) and Using (1.2.2) we derive

$$\begin{aligned} g(h(PZ,X),FW) &= -g(PZ,\overline{\nabla}_X JW + PW) \\ &= -g(PZ,\overline{\nabla}_X JW) + g(PZ,\overline{\nabla}_X PW) \\ &= g(JPZ,\overline{\nabla}_X W) + g(PZ,\overline{\nabla}_X PW) \\ &= g(P^2Z,\overline{\nabla}_X W) + g(FPZ,h(X,W)) + g(PZ,\overline{\nabla} \end{aligned}$$

Then from lemma 4.2.1 and 5.2.1 we obtain

$$\begin{aligned} g(h(PZ,X),FW) &= -cos^2\theta X(\ln f)g(Z,W)+g(FPZ,h(X,W)), \\ &\quad +X(\ln f)g(PZ,PW). \end{aligned}$$

Also from (1.4.13) we get

$$\begin{aligned} g(h(PZ,X),FW) &= -cos^2\theta X(\ln f)g(Z,W)+g(FPZ,h(X,W)), \\ &\quad +X(\ln f)cos^2\theta g(Z,W). \end{aligned}$$

Thus for $Z = W$ we have

$$g(h(PZ,X),FZ) = g(FPZ,h(X,Z)),$$

for X in $T(M_T)$ and Z in $T(M_\theta)$.

Theorem 5.2.2 [36]. Let $\overline{M}$ be a Kaehler manifold. Then there do not exist warped-product submanifolds $M = M_T \times_f M_\theta$ in M such that M_T is a holomorphic submanifold and M_θ is a proper slant submanifold of M.

Proof. For $X \in T(M_T)$ and $Z \in T(M_\theta)$, from Lemma 4.2.1 we have

$$g(\overline{\nabla}_{PZ}X,Z) = X(\ln f)g(PZ,Z) = 0.$$

Since

$$g(PZ,Z) = 0.$$

Now from Lemma 5.2.3

$$X(\ln f)cos^2\theta g(Z,Z) = -g(h(PZ,X),FZ), \tag{5.2.11}$$

$$g(h(Z,X),FPZ) = X(\ln f)cos^2\theta g(Z,Z), \tag{5.2.12}$$

for X in $T(M_T)$ and Z in $T(M_\theta)$. On the other hand, from Lemma 5.2.3

$$g(h(PZ,X),FZ) = g(FPZ,h(X,Z)), \tag{5.2.13}$$

for X in TM_T and Z in TM_θ. Thus from (5.2.11), (5.2.12) and (5.2.13) we get

$$X(\ln f)cos^2\theta g(Z,Z) = 0.$$

Since M_θ is a proper slant and Z is non null, we obtain

$$X(\ln f) = 0$$

This proves our assertion.

Following example of a CR-warped product submanifold of a Kaehler manifold shows that Theorem 5.2.2 is not true for CR-warped product submanifolds in Kaehler manifolds.

Example(5.2.1): Consider in R^8 the submanifold M given by the equations

$$x_1 = cos\theta,\ x_2 = scos\theta,\ x_3 = tcos\varphi,\ x_4 = scos\varphi.$$

$$x_5 = tsin\theta,\ x_6 = ssin\theta,\ x_7 = tsin\varphi,\ x_8 = ssin\varphi,$$

$$\theta, \varphi \in (0, \pi/2).$$

Then TM is spanned by $Z_t, Z_s, Z_\theta, Z_\varphi$, where

$$Z_t = cos\theta\partial x_1 + cos\varphi\partial x_3 + sin\theta\partial x_5 + sin\varphi\partial x_7.$$

$$Z_s = cos\theta\partial x_2 + cos\varphi\partial x_4 + sin\theta\partial x_6 + sin\varphi\partial x_8.$$

$$Z_\theta = tsin\theta\partial x_1 + ssin\theta\partial x_2 + tcos\theta\partial x_5 + scos\theta\partial x_6.$$

$$Z_\varphi = tsin\varphi\partial x_3 + ssin\varphi\partial x_4 + tcos\varphi\partial x_7 + scos\varphi\partial x_8.$$

We obtain that $D = span\{Z_t, Z_s\}$ is invariant with respect to J. Moreover, JZ_θ and JZ_φ are orthogonal to TM. Hence $D^\perp = span\{Z_\theta, Z_\varphi\}$ is anti-invariant with respect to J. Thus M is a CR-submanifold of R_8. Furthermore, we can derive that D and $D^\perp$ are integrable. Denoting the integral manifolds of D and $D^\perp$ by M_T and $M_\perp$, respectively, then the induced metric tensor is

$$\begin{aligned} g &= 2dt^2 + 2ds^2 + (t^2 + s^2)(d\theta^2 + d\varphi^2) \\ &= g_{M_T} + (t^2 + s^2)g_{M_\perp}. \end{aligned}$$

Thus M is a CR-warped product submanifold of R^8 with warping function $f = \sqrt{t}^2 + s^2$.

5.3 Warped Product CR-Submanifolds $N_\perp \times_f N_T$ of Nearly Kaehler Manifolds.

Throughout this section, we assume that $\bar{M}$ is a nearly Kaehler manifold and $M = N_\perp \times_f N_T$ is a warped product CR-submanifold of $\bar{M}$. In the following sections we shall give and prove recent results by V. A. Khan, K. A. Khan *et. al* [28] which extend the study of warped product CR-submanifolds.

The nearly Kaehler structure on an almost Hermitian manifold $\bar{M}$ can be characterized by

$$(a)\ \ \mathcal{P}_U V + \mathcal{P}_V U = 0 \qquad (b)\ \ Q_U V + Q_V U = 0, \tag{5.3.1}$$

for each $U,\ V \in T(M)$, where P and Q denotes the tangential and normal components of ∇J. On the submanifold M of $\bar{M}$, by property (p4) mentioned in chapter two of P and Q, we also have

$$\mathcal{P}_X JX + Q_X JX = 0, \tag{5.3.2}$$

for each $X \in T(N_T)$.

By Corollary 4.2.1 $N_\perp$ is totally geodesic in M and N_T is totally umbilical in M. Thus, if h^T and $\bar{h}$ denote the second fundamental forms of the immersions of N_T in M and in $\bar{M}$ respectively, then

$$\hat{h}(X,Y) = h^T(X,Y) + h(X,Y), \tag{5.3.3}$$

$$h^T(X,Y) = -g(X,Y)grad(\ln f), \tag{5.3.4}$$

for each $X,\ Y \in T(N_T)$.

The Lemma 4.2.1 can be restated as

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X. \tag{5.3.5}$$

Hence,

$$g(\nabla_X Z, X) = (Z \ln f)\|X\|^2 = g(\nabla_{JX} Z, JX). \tag{5.3.6}$$

On applying formulae (1.3.2), (5.3.1) and (5.3.2), equation (5.3.6) can be written as

$$(Z \ln f)\|X\|^2 = g(JZ, h(X, JX)). \tag{5.3.7}$$

Replacing X by JX in equation (5.3.7), we get

$$(Z \ln f)\|X\|^2 = -g(JZ, h(X, JX)). \tag{5.3.8}$$

From equations (5.3.7) and (5.3.8),

$$(Z \ln f)\|X\|^2 = 0. \tag{5.3.9}$$

If M is assumed to be a proper warped product CR-submanifold, then $Z(\ln f) = 0$ i.e., M is simply a CR-product. In other words, the Theorem of B. Y. Chen [16] is extended to the setting of nearly Kaehler manifold as

Theorem 5.3.1 [28] There does not exist a proper warped product CR-submanifold $N_\perp \times_f N_T$ in nearly Kaehler manifolds.

5.4 Warped Product CR-Submanifolds $N_T \times_f N_\perp$ of Nearly Kaehler Manifolds.

In this section we shall study the warped product CR-submanifolds of the type $N_T \times \ _f N_\perp$ in a nearly Kaehler manifold $\bar{M}$. First, we shall prove [28]

Lemma 5.4.1 . Let M be a warped product CR-submanifold of a nearly Kaehler manifold $\bar{M}$. Then we have

(i) $\quad g(h(X, Y), JZ) = 0$

(ii) $\quad g(\nabla_Z X, W) = X(\ln f) g(Z, W) = g(h(JX, Z), JW)$

for each $X,\ Y \in T(N_T)$ and $Z,\ W \in T(N_\perp)$.

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Proof. By equations (2.2.6) and (2.2.7),

$$g(A_{FZ}X, Y) = g(\nabla_X Z, JY) - g(\mathcal{P}_X Z, Y).$$

The first term in the right hand side of the above equation is zero in view of Lemma 4.2.1. Thus, the equation reduces to

$$g(A_{FZ}X, Y) = -g(\mathcal{P}_X Z, Y).$$

The left hand side of the above equation is symmetric in X and Y whereas the right hand side is skew symmetric in X and Y . That proves

$$g(h(X,Y), JZ) = g(\mathcal{P}_X Z, Y) = 0.$$

The first equality in (ii) is an immediate consequence of Lemma 4.2.2 (ii). For the second equality, by Gauss formula, we may write

$$\begin{aligned} g(h(JX,Z), JW) &= g(\bar{\nabla}_Z JX, JW \\ &= g(Q_Z X, JW) + g(\nabla_Z X, W) \\ &= g(Q_Z JX, W) + X(\ln f)g(Z,W). \\ &= -g(\mathcal{P}_Z W, JX) + X(\ln f)g(Z,W). \end{aligned}$$

The first term in the right hand side of the above equation is zero by virtue of (3.2.16) and the equation reduces to

$$g(h(JX,Z), JW) = (X \ln f)g(Z,W)$$

which completes the proof of statement (ii).

Theorem 5.4.1 [28]. Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ with integrable distributions D and $D^{\perp}$. Then M is locally a CR-warped product if and only if

$$A_{JZ}X = -(JX\mu)Z \tag{5.4.1}$$

for each $X \in D$, $Z \in D^\perp$ and μ, a C^∞-function on M such that $W\mu = 0$ for each $W \in D^\perp$.

Proof. If M is a warped product CR-submanifold $N_T \times {}_fN_\perp$, then on applying Lemma 5.4.1, we obtain equation (5.4.1). In this case $\mu = \ln f$.

Conversely, suppose $A_{JZ}X = -(JX\mu)Z$, then

$$g(h(X,Y), JZ) = 0$$

i.e., $h(X,Y) \in \mu$, for each X, $Y \in D$. As D is assumed to be integrable, by (3.2.14), $Q_XY = 0$ and therefore by formula (2.2.7)

$$F\nabla_XY = h(X, JY) - fh(X,Y).$$

As $h(X,Y) \in \mu$ for each X, $Y \in D$, $FU \in JD^\perp$ for each $U \in TM$ and $f\xi \in \mu$ for all $\xi \in TM^\perp$, we deduce from the above equation that $\nabla_XY \in D$. That means, leaves of D are totally geodesic in M. Now,

$$\begin{aligned} g(\nabla_ZW, X) &= g(J\bar{\nabla}_ZW, JX) \\ &= -g(\mathcal{P}_ZW, JX) - g(A_{JW}Z, JX). \end{aligned}$$

The first term in the right hand side of the above equation vanishes in view of (3.2.16) and the second term on making use of (5.4.1) reduces to $X\mu\, g(Z,W)$. That is, we have

$$g(\nabla_ZW, X) = X\mu\, g(Z,W). \tag{5.4.2}$$

Now, by Gauss formula

$$g(h^\perp(Z,W), X) = g(\nabla_ZW, X)$$

where $h^\perp$ denotes the second fundamental form of the immersion of $N_\perp$ into M. Using (5.4.2), the last equation gives

$$g(h^\perp(Z,W), X) = X\mu\, g(Z,W)$$

which shows that each leaf $N_\perp$ of $D^\perp$ is totally umbilical in M. Moreover, the fact that $W\mu = 0$, for all $W \in D^\perp$, implies that the mean curvature vector on $N_\perp$ is parallel along $N_\perp$ i.e., each leaf of $D^\perp$ is an extrinsic sphere in M. Thus M is locally a warped product $N_T \times {}_f N_\perp$ of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of M [11]. Here N_T is a leaf of D and $N_\perp$ is a leaf of $D^\perp$ and f is a warping function.

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